Math 3 Variable Manipulation Part 8
Exponential & Log Functions

LOGARITHMIC EQUATIONS (LOGS)
Exponential and logarithmic functions are inverses of each other. Two of the most widely used logarithms are the common log, which is base 10, and is written as $\log_{10} x = \log x$ and the natural log, which is base e, and is written as $\log_e x = \ln x$.

Definition of a Logarithm

$\log_b x = c$ if and only if $b^c = x$
$\ln_x c =$ if and only if $e^c = x$

y* = b*^x ....is equivalent to.............. ***log*$_b$(*y*) = *x***
(means the exact same thing as)

Example: Solve $log_2(x) = 4$
Solution: Solve by using the Relationship:
$log_2(x) = 4$
$2^4 = x =$ **16 = *x***

Example: Solve $log_2(8) = x$
Solution: Convert the logarithmic statement into its equivalent exponential form, using The Relationship:
$log_2(8) = x$
$2^x = 8$
But $8 = 2^3$, so:
$2^x = 2^3$ ***x* = 3**

Note that this could also have been solved by working directly from the definition of a logarithm: What power, when put on "2", would give you an 8? The power 3, of course!

Sample Questions:

1. Rewrite each of the following in exponential form.

a. $\log_4 64 = 3$ b. $\log_5 \frac{1}{25} = -2$ c. $\log_{65} 1 = 0$

2. Rewrite each of the following in logarithmic form.

a. $3^4 = 81$ b. $10^{-2} = \frac{1}{100}$ c. $6^1 = 6$

3. Use the properties of logarithms to evaluate the expression without a calculator.

a. $\log 10^{-4}$ b. $e^{\ln 6}$ c. $\log_3 1$ d. $\log_{50} 50$

Rewrite each of the equations in exponential form.

4. $\log_4 1 = 0$

5. $\log_3 243 = 5$

6. $\log_7 \frac{1}{343} = -3$

7. $\log_6 216 = 3$

Rewrite each of the equations in logarithmic form.

8. $6^{-1} = \frac{1}{6}$

9. $9^3 = 729$

10. $7^1 = 7$

11. $3^3 = 27$

NATURAL LOGARITHMIC EQUATIONS (LN)

Remember, exponential and logarithmic functions are inverses of each other. The last two sections have dealt with logs with many bases. If it does not say exactly, it is assumed base 10, and is written as $\log_{10} x = \log x$. The other common is the natural log, which is base e, and is written as $\log_e x = \ln x$.

Definition of a Logarithm

$$\log_b x = c \text{ if and only if } b^c = x$$
$$\ln_x c = \text{ if and only if } e^c = x$$

Example: Solve $ln(x) = 3$
Solution: The base of the natural logarithm is the number "*e*" (with a value of about 2.7). To solve this, I'll use The Relationship, and I'll keep in mind that the base is "*e*":

$ln(x) = 3$
$e^3 = x$ or solve for x with calculator: x = 20.086, rounded to three decimal places.

Example: Solve $log_2(x) = 4.5$
Solution: First, convert the log equation to the corresponding exponential form, using the relationship:

$log_2(x) = 4.5$
$2^{4.5} = x$

This requires a calculator for finding the approximate decimal value.
The answer is $x = 2^{4.5} = 22.63$, rounded to two decimal places.

Example: Solve $log_2(3x) = 4.5$
Solution: $log_2(3x) = 4.5$

$2^{4.5} = 3x$
$(2^{4.5}) \div 3 = x$
$x = 7.54247233266...$

The answer is $x = (2^{4.5}) / 3$, or $x = 7.54$, rounded to two decimal places.

Sample Questions:
Evaluate the expression without a calculator.

12. $\ln 1$

13. $e^{\ln x2}$

14. $\ln e^{10x+5}$

THE PRINCIPLE OF EXPONENTIAL EQUALITY
For any real number b, where $b \neq -1, 0$, or 1, $b^{x1} = b^{x2}$ is equivalent to $x_1 = x_2$. In other words, powers of the same base are equal if and only if the exponents are equal.

Example: Solve the following.

a. $\log_2 8 = x$ b. $\log_7 x = 2$ c. $\log_x 125 = 3$

Solution:

a. $2^x = 8$ $2^x = 2^3$ $x = 3$	Rewrite in exponential form and solve for x.
b. $7^2 = x$ $49 = x$	Rewrite in exponential form and solve for x.
c. $x^3 = 125$ $x^3 = 5^3$ $x = 5$	Rewrite in exponential form and solve for x.

Example: Solve the following.

a. $27^2 = 9^{x+1}$ b. $\log_4(3x-2) = 2$

Solution:

a. $27^2 = 9^{x+1}$ $(3^3)^2 = (3^2)^{x+1}$ $3^6 = 3^{2x+2}$ $6 = 2x+2$ $4 = 2x$ $2 = x$	Change the bases on both sides of the equation so that they are the same base. Rewrite using exponent rules. Simplify using exponent rules. Solve using the principle of exponential equality.
b. $\log_4(3x-2) = 2$ $4^2 = 3x-2$ $16 = 3x-2$ $18 = 3x$ $6 = x$	Rewrite in exponential form and solve for x.

MORE RULES OF LOGARITHMS

Product Rule

$\log_b(xy) = \log_b x + \log_b y$ $\ln(xy) = \ln x + \ln y$

Quotient Rule

$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$ $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

Power Rule

$\log_b(x)^c = c\log_b x$ $\ln(x)^c = c\ln x$ For each of these rules, $b \neq 1$, x, y, and c are real numbers.

Example: Expand the following expressions.

a. $\log\frac{a^4 b}{c^5}$ b. $\ln\sqrt{m^3 n}$ c. $\log\frac{2w^4h^3}{a^2b^5}$

Solution:

a. $\log\frac{a^4b}{c^5}$	
$\log a^4 + \log b - \log c^5$	Use the product and quotient rules to rewrite the expression.
$4\log a + \log b - 5\log c$	Use the power rule to rewrite the expression.
b. $\ln\sqrt{m^3n}$	
$\frac{1}{2}\ln m^3 n$	Use the power rule to rewrite the expression.
$\frac{1}{2}\left(\ln m^3 + \ln n\right)$	Use the product rule to rewrite the epxression.
$\frac{1}{2}\left(3\ln m + \ln n\right)$	Use the power rule to rewrite the expression.
$\frac{3}{2}\ln m + \frac{1}{2}\ln n$	Use the distributive property.
c. $\log\frac{2w^4h^3}{a^2b^5}$	
$\log 2 + \log w^4 + \log h^3 - \left(\log a^2 + \log b^5\right)$	Use the product and the quotient rules to rewrite the expression.
$\log 2 + 4\log w + 3\log h - \left(2\log a + 5\log b\right)$	Use the power rule to rewrite the expression.
$\log 2 + 4\log w + 3\log h - 2\log a - 5\log b$	Use the distributive property.

Example: Condense the following expressions.

a. $\ln(x+1) - 3\ln(x-2)$ b. $\log 3 + 4\log a - \frac{2}{3}\log b$ c. $4\ln a - 3\ln b + 7\ln c - 5\ln(d+1)$

Solution:

a. $\ln(x+1) - 3\ln(x-2)$	
$\ln(x+1) - \ln(x-2)^3$	Use the power rule to rewrite the expression.
$\ln\frac{x+1}{(x-2)^3}$	Use the quotient rule to rewrite the expression.
b. $\log 3 + 4\log a - \frac{2}{3}\log b$	
$\log 3 + \log a^4 - \log b^{2/3}$	Use the power rule to rewrite the expression.
$\log\frac{3a^4}{\sqrt[3]{b^2}}$	Use the product and quotient rules to rewrite the expression

Example: Solve $log_2(x) + log_2(x - 2) = 3$

Solution: Use log rules to combine the two terms on the left-hand side of the equation:

$log_2(x) + log_2(x - 2) = 3$
$log_2((x)(x - 2)) = 3$
$log_2(x^2 - 2x) = 3$
$log_2(x^2 - 2x) = 3$
$2^3 = x^2 - 2x$
$8 = x^2 - 2x$
$0 = x^2 - 2x - 8$
$0 = (x - 4)(x + 2)$
$x = 4, -2$

But if $x = -2$, then "$log_2(x)$", from the original logarithmic equation, will have a negative number for its argument (as will the term "$log_2(x - 2)$"). Since logs cannot have zero or negative arguments, then the solution to the original equation cannot be $x = -2$. **The solution is $x = 4$.**

Example: Solve $log_2(log_2(x)) = 1$

Solution: This may look overly-complicated, but just apply the relationship twice:

$log_2(log_2(x)) = 1$
$2^1 = log_2(x)$
$2 = log_2(x)$
$x = 2^2 = 4$ Then the solution is $\boldsymbol{x = 4}$

Example: Solve $log_2(x^2) = (log_2(x))^2$.

Solution: First, I'll write out the square on the right-hand side:

$log_2(x^2) = (log_2(x))^2$
$log_2(x^2) = (log_2(x))\,(log_2(x))$

Apply the log rule to move the "squared", from inside the log on the left-hand side of the equation, out in front of that log as a multiplier. Then I'll move that term to the right-hand side:

$2log_2(x) = [log_2(x)]\,[log_2(x)]$
$0 = [log_2(x)]\,[log_2(x)] - 2log_2(x)$
$0 = [log_2(x)]\,[log_2(x) - 2]$
$log_2(x) = 0$ or $log_2(x) - 2 = 0$
$2^0 = x$ or $log_2(x) = 2$
$1 = x$ or $2^2 = x$ $1 = x$ or $4 = x$ **The solution is $x = 1, 4$**

Example: Solve $ln(e^x) = ln(e^3) + ln(e^5)$.

Solution: $ln(e^x) = x$. Similarly, $ln(e^3) = 3$ and $ln(e^5) = 5$. So the given equation simplifies quite nicely:

$ln(e^x) = ln(e^3) + ln(e^5)$
$x = 3 + 5 = 8$ The solution is $\boldsymbol{x = 8}$.

Note: This could also have been solved using log rules:

$ln(e^x) = ln(e^3) + ln(e^5)$
$ln(e^x) = ln((e^3)(e^5))$
$ln(e^x) = ln(e^{3+5})$
$ln(e^x) = ln(e^8)$

Comparing the arguments:

$e^x = e^8$
$\boldsymbol{x = 8}$

Sample Questions:
Solve the following

15. $\log_7 x = 3$

16. $\log_5 625 = x$

17. $\log_9 x = 0$

18. $\log_3 \frac{1}{243} = x$

19. $4^3 = 8^{3-x}$

20. $16^{x+1} = 8^{x+3}$

21. $(\frac{1}{9})^{2-x} = 27^2$

22. $\log_2(17x - 2) = 5$

23. $\log_4(3x - 5) = 3$

24. $-2\log_5 7x = 2$

25. $-6\log_3 (x - 3) = -24$

26. $\log_9 9^3$

27. $4^{\log_4 (2x)}$

28. $\log_6 1$

29. $\log_9 9$

30. $\ln (n^2 + 12) = \ln (-9n - 2)$

31. If $\log_4 x = 3$, then x = ?

32. What is the value of $\log_4 64$?

33. Which of the following is a value of n that satisfies $\log_n 64 = 2$?

a. 4
b. 6
c. 8
d. 12
e. 32

34. What is the value of $\log_2 = 8$?

35. What is the real value of x in the equation $\log_2 24 - \log_2 3 = \log_5 x$?

36. If $\log_a x = n$ and $\log_a y = p$, then $\log_a (xy)2 = ?$

a. np
b. 2np
c. 4np
d. n + p
e. 2(n + p)

37. Which of the following ranges of consecutive integers contains the value of the expression $\log 9(9^{\frac{7}{3}})$?

f. 0 and 1
g. 1 and 2
h. 2 and 3
i. 5 and 6
j. 7 and 8

Answer Key

1.

a. $4^3 = 64$	The base is 4 and the exponent is 3.
b. $5^{-2} = \frac{1}{25}$	The base is 5 and the exponent is -2.
c. $65^0 = 1$	The base is 65 and exponent is 0.

2.

a. $\log_3 81 = 4$	The base is 3 and the exponent is 4.
b. $\log_{10} \frac{1}{100} = -2$	The base is 10 and the exponent is -2.
c. $\log_6 6 = 1$	The base is 6 and exponent is 1.

3.

a. $\log 10^{-4} = -4$	Use of basic property 3.
b. $e^{\ln 6} = 6$	Use of basic property 4.
c. $\log_3 1 = 0$	Use of basic property 1.
d. $\log_{50} 50 = 1$	Use of basic property 2.

4. .
5. .
6. .
7. .
8. .
9. .
10. .
11. .
12. .
13. .
14. .
15. .
16. .
17. .
18. .
19. .
20. .
21. .
22. .
23. .
24. 1/35
25. 84
26. .
27. .
28. ..
29. .
30. $-2, -7$
31. 64
32. 3
33. C
34. 3
35. 125
36. E
37. C