

Math 3 Proportion & Probability Part 1

Percent, Ratio, Proportion, Rate, Average

Patterns, Combinations & Probability

MATH 1 LEVEL REVIEW

PERCENT/RATE/PROPORTION

Sample Questions:

1. If 20% of a number is 125, what is 28% of the number?

2. In the process of milling grain, 3% of the original is lost because of spillage, and another 5% of the original is lost because of mildew. If the mill starts out with 490 tons of grain, how much (in tons) remains to be sold after milling?

3. What is $\frac{1}{4}$ of 20% of \$50,000?

4. At Paradigm, students must take both a written exam and an oral exam. In the past, 90% of the students passed the written exam and 75% of those who passed the written exam also passed the oral exam. Based on these figures, about how many students in a random group of 350 students would you expect to pass both exams?

5. The original price of a pair of shoes is \$39.99. They are on sale 40% off and the tax rate is 6%. What is the final price?

6. Cost of dinner is \$29.95. You want to pay 20% tip and tax rate is 2%. What is the total cost of dinner?

7. Which is the best buy? 6 shirts for \$25.50, 4 shirts for \$18.00 or 5 shirts for \$21

8. Lauren took 12 hours to read a 360 page book. At this rate, how long will it take her to read a 400 page book?

AVERAGE FORMULA

To find the average of a set of numbers, add them up and divide by the number of numbers.

$$\text{Average} = \frac{\text{Sum of the terms}}{\text{Number of the terms}}$$

Sample Questions:

9. What is the average of $2/3$ and $4/5$?
10. What is the average of the expressions $x + 4$, $2x - 5$, $2x + 1$?
11. Tracy mowed lawns for 2 hours and earned \$7.40 per hour. Then she washed windows for 3 hours and earned \$6.50 per hour. What were Tracy's average earnings per hour for all 5 hours?
12. After taking 3 quizzes, your average is 72 out of 100. What must your average be on the next two quizzes so that on 5 quizzes you increase your average to 77?
13. If the average of 8, 11, 25, and p is 15, find p.
14. If $a = 3b = 6c$, what is the average of a, b and c in terms of a?

WEIGHTED AVERAGE

Another type of average problem involves the weighted average - which is the average of two or more terms that do not all have the same number of members. To find the weighted term, multiply each term by its weighting factor, which is the number of times each term occurs. Use weighted average to calculate GPA or figure out mixtures.

The formula for weighted average is:

$$\text{Weighted Average} = \frac{\text{Sum of Weighted Terms}}{\text{Total Number of Terms}}$$

Be careful! You will get the wrong answer if you add the two average scores and divide the answer by two.

Sample Questions:

15. Fifteen accounting majors have an average grade of 90. Seven marketing majors averaged 85, and ten finance majors averaged 93. What is the weighted mean for the 32 students?
16. 5 fl. oz. of a 2% alcohol solution was mixed with 11 fl. oz. of a 66% alcohol solution. Find the concentration of the new mixture.
17. James must have a 3.0 GPA to be able to play on the team. He already has a B-, B-, C, B-, A, A, B-. What is the minimum grade he needs to get in his last class to be able to play on the team?
- | | | | |
|----|-----|----|-----|
| A | 4.0 | C | 2.0 |
| A- | 3.7 | C- | 1.7 |
| B+ | 3.3 | D+ | 1.3 |
| B | 3.0 | D | 1.0 |
| B- | 2.7 | F | 0.0 |
| C+ | 2.3 | | |

AVERAGE RATE: DISTANCE RATE PROBLEMS

Average rate is its own special concept and you will notice that it is NOT an Average of the Speeds (which would be something like the Sum of the Speeds / the Number of Different Speeds or what we know as the Arithmetic Mean). Average Rate is completely different. The most common rate is speed-distance over time-and the most common question about average rates is average speed-total distance over total time.

$$\text{Average Rate} = \frac{\text{Total Distance}}{\text{Total Time}}$$

Sample Questions:

18. Robert rides his scooter an average of 5 miles per hour, and Cindy rides her roller blades an average of 6 miles per hour. At these rates, how much longer does it take Runner A than Runner B to travel 3 miles?
19. A cargo plane flew to the maintenance facility and back. It took one hour less time to get there than it did to get back. The average speed on the trip there was 220 mph. The average speed on the way back was 200 mph. How many hours did the trip there take?

AVERAGE RATE: WORK RATE PROBLEMS

To work out these problems, it is usually assumed that a work-agent (say a man) takes certain number of time units T (usually days or hours) to complete the work. So the work rate of the agent in one time unit (a day or an hour) is expressed as $\frac{1}{T}$ th portion of the total amount of work. The reason why this work rate in terms of work portion per unit time is the most important concept in Time and Work problems is - it makes possible summing up of efforts of more than one type of work agents working together at different work rates over unit time. This is the core concept behind the deductions of Time and Work problem of any type.

Example: A job can be completed by 4 men in 24 days and 4 women in 12 days. In how many days would the 4 men and 4 women working together complete the work?

Solution: In case of 4 men and 4 women working independently completing a job in 24 days and 12 days respectively, if we are asked to find the number of days taken to complete the job by 4 men and 4 women working together, by the conventional approach, we derive the per day work rate of 1 man and 1 woman as,

$$\frac{1}{4 \times 24} = \frac{1}{96} \text{ portion of work, and } \frac{1}{4 \times 12} = \frac{1}{48} \text{ portion of work.}$$

When these two teams work together for 1 day, we would now be able to sum up their efforts in one day as, $\frac{4}{96} + \frac{4}{48} = \frac{3}{24} = \frac{1}{8}$ portion of work.

We now arrive at the desired result using unitary method. The number of days that the two teams would take to complete the job working together would just be inverse of the per day work portion, that is, 8 number of days. This approach seems to be a bit complex as it deals with inverses, but this is the usual method followed.

Sample Questions:

20. It takes Trevon ten hours to clean an attic. Cody can clean the same attic in seven hours. Find how long it would take them if they worked together.
21. Working alone, Carlos can oil the lanes in a bowling alley in five hours. Jenny can oil the same lanes in nine hours. If they worked together how long would it take them?

COUNTING THE POSSIBILITIES

The fundamental counting principle: If there are **m ways** one event can happen and **n ways** a second event can happen, then there are **$m \times n$ ways** for the two events to happen. For example, with 7 shirts and 5 pairs of pants to choose from, you can put together $7 \times 5 = 35$ different outfits. For more items, multiply all the ways together. For example, with 7 shirts, 5 pairs of pants, 8 ties and 2 pairs of shoes to choose from, you can put together $7 \times 5 \times 8 \times 2 = 560$ different outfits.

COMBINATIONS: THE SLOT METHOD

Combination problems ask you how many different ways a number of things could be chosen or combined. First, figure out the number of slots you need to fill. Then fill in those slots. Finally, find the product.

Example: At the school cafeteria, students can choose from 3 different salads, 5 different main dishes, and 2 different desserts. If Isabel chooses one salad, one main dish, and one dessert for lunch, how many different lunches could she choose?

Solution: There are three slots to fill, one for each item: salad, main dish, dessert. And the number of possibilities for each is pretty clear. Set up the slots and take the produce as your answer.

$$\begin{array}{r} 3 \\ \hline \text{Salad} \end{array} \quad \times \quad \begin{array}{r} 5 \\ \hline \text{Main} \end{array} \quad \times \quad \begin{array}{r} 2 \\ \hline \text{Dessert} \end{array} \quad = 30$$

Sample Questions:

22. Reggie knows how to make 5 different entrees, 4 different side dishes, and 6 different desserts. How many distinct complete meals, each consisting of an entrée, a side dish, and a dessert, can Reggie make?
23. Chris owns 10 different dress shirts, 4 different pairs of pants, and 7 different ties. How many distinct outfits, each consisting of a shirt, a pair of pants, and a tie, can Justin make?

PROBABILITY

Probability is always the number of desired outcomes divided by the number of possible outcomes. It can be expressed as a fraction, or sometimes, as a decimal.

$$\text{Probability} = \frac{\text{Favorable outcomes}}{\text{Total possible outcomes}}$$

Sample Questions:

24. There are 15 balls in a box: 7 balls are green, 3 are blue and 5 are white. Then 1 green and 1 blue balls are taken from the box and put away. What is the probability that a blue ball will NOT be selected at random from the box?
25. Seth has 4 plaid shirts and 5 solid-colored shirts hanging together in a closet. In his haste to get ready for work, he randomly grabs 1 of these 9 shirts. What is the probability that the shirt Seth grabs is plaid?
26. A partial deck of cards was found sitting out on a table. If the partial deck consists of 6 spades, 3 hearts, and 7 diamonds, what is the probability of randomly selecting a red card from this partial deck?

MATH 2 REVIEW

COMBINED PERCENT INCREASE AND DECREASE

Example: A price went up 10% one year, and the new price went up 20% the next year. What was the combined percent increase?

Solution: First year: $100 + (10\% \text{ of } 100) = 110$. Second year: $110 + (20\% \text{ of } 110) = 132$. That's a combined 32% increase.

Sample Questions:

27. On Friday, a computer was priced at \$800. On the following Wednesday, the price was reduced by 15%. On the following Friday, the price was further reduced by 20%. What percent of the original price was the final price?
28. A shoe store charges \$39 for a certain type of sneaker. This price is 40% more than the amount it costs the shoe store to buy one pair of these sneakers. At an end-of-the-year sale, sales associates can purchase any remaining sneakers at 20% off the shoe store's cost. How much would it cost an employee to purchase a pair of sneakers of this type during the sale (excluding sales tax)?

PART-TO-PART AND PART-TO-WHOLE RATIOS

A part-to-part ratio can be turned into two part-to-whole ratios by putting **each number in the original ratio over the sum of the parts**. If the ratio of males to females is 1 to 2, then the males-to-people ratio is 1 to 3 and the females-to-people ratio is 2 to 3.

Sample Questions:

29. Esther is making $2 \frac{1}{4}$ gallons of punch for a large party. While mixing the punch, she uses $\frac{1}{2}$ gallon of pineapple juice. What fraction of the punch consists of pineapple juice?
30. In a poll, 44 people were in favor of constructing a new high school, 58 were against it, and 8 people had no opinion. What fraction of those people polled were in favor of constructing a new high school?

MULTI STEP RATIO PROBLEMS

Example: A room is 16 feet, 8 inches long, and the ratio of the length to the width is 4 to 5. What is the width of the room?

Solution: Step 1: Since the length is given in both feet and inches, let's convert it to inches using the fact that 1 foot equals 12 inches. To find how many inches are in 16 feet, we multiply 16 feet by 12 inches:

$$16 \text{ feet, 8 inches} = (16 \times 12) + 8 = 192 + 8 = 200$$

We found that the length is 200 inches.

Step 2: Let x represent the width. We can now set up the equation:

$$\frac{\text{width}}{\text{length}} = \frac{4}{5} = \frac{x}{200}; \text{ The width is 160 inches.}$$

Let's now convert inches to feet so that the units for the width are consistent with the units for the length. Since 1 foot is 12 inches, we divide 160 inches by 12 to find out how many feet are in 160 inches:

$160:12 = 13$, with the remainder of 4 inches. The width is 13 feet, 4 inches. The room width is 13 feet, 4 inches.

Sample Questions:

31. At a small college, the ratio of men to women is 9:4. If there are presently 720 women, how many men are at the college?
32. How many additional women would it take to reduce the ratio of men to women to 2:1?

33. If $x:y = 5:2$ and $y:z = 3:2$, what is the ratio of $x:z$?

34. A school has 300 students. If the ratio of boys to girls is 31 to 44, how many more girls are there in the school?

RATE PROBLEM USING VARIABLE MANIPULATION

Sample Problems:

35. A car and a bus set out at 2 p.m. from the same point, headed in the same direction. The average speed of the car is 30 mph slower than twice the speed of the bus. In two hours, the car is 20 miles ahead of the bus. Find the rate of the car.
36. A dog eats 7 cans of food in 3 days. At this rate, how many cans of food does the dog eat in $3 + d$ days?

37. Harry is a piano student who can learn 2 new pieces of music in a week. If his piano teacher gives him 3 new pieces every week for 4 weeks, how many weeks will it take Harry to learn all these pieces?

38. A hot-air balloon 70 meters above the ground is falling at a constant rate of 60 meters per second while another hot-air balloon 10 meters above the ground is rising at a constant rate of 15 meters per second. To the nearest tenth of a second, after how many seconds will the 2 balloons be the same height above the ground?

AVERAGE RATE

Average rate is *not* simply the average of the rates. It's the average of the total amounts. The most common rate is speed-distance over time-and the most common question about average rates is average speed-total distance over total time. **Remember:** You can add distances and you can add times, but you cannot add rates. Think about it: If you drive 20 mph on one street, and 40 mph on another street, does that mean you averaged 60 mph?

Sample Questions:

39. If the first 120 miles of a journey is at 40 mph and the next 120 miles is at 60 mph, what is the average speed?

40. Tracey ran to the top of a steep hill at an average pace of 6 miles per hour. She took the exact same trail back down. To her relief, the descent was much faster; her average speed rose to 14 miles per hour. If the entire run took Tracey exactly one hour to complete and she did not make any stops, what is the length of trail in miles one way?

THE WEIGHTED AVERAGE

Regular Average Example: If Sally received a grade of 90 on a test last week and a grade of 100 on a test this week, what is her average for the two tests? Piece of cake, right? The answer is 95. You added the scores and divided by 2. Now let's turn the same question into a weighted average question.

Weighted Average Example: If Sally's average for the entire year last year was 90, and her average for the entire year this year was 100, is her average for the two years combined equal to 95? The answer is "not necessarily." If Sally took the same number of courses in both years, then yes, her average is 95. But what if last year she took 6 courses while this year she took only 2 courses? Can you compare the two years equally? ACT likes to test your answer to this question.

Example: An online seed supplier packages a seed mix that costs the company \$30.30 per pound. The mix includes poppy seeds costing \$35.65 per pound and clover seeds costing \$8.90 per pound. If a worker is going to prepare some of this mix and has already measured out 15 pounds of poppy seeds, what quantity of clover seeds should he add? *Write your answer as a whole number or as a decimal rounded to the nearest tenth.*

Solution: First, write an equation.

Total price of poppy seeds plus total price of clover seeds equals total price of seed mix.

$$35.65(15) + 8.9x = 30.3(15 + x)$$

Solve the equation

$$35.65(15) + 8.9x = 30.3(15 + x)$$

$$534.75 + 8.9x = 454.5 + 30.3x$$

$$534.75 - 21.4x = 454.5$$

$$-21.4 = -80.25$$

$$X \approx 3.8$$

The worker should add about 3.8 pounds of clover seeds.

Sample Questions:

41. A factory is using eugenol, a compound extracted from cinnamon and cloves, in some of its products.

The factory has just 15 gallons of oil that contains 84% eugenol, in addition to plenty of oil containing 59% eugenol. How many gallons of the oil with 59% eugenol should be added to the oil with 84% to obtain a product with 74% eugenol?

42. A candy maker is mixing up some sugar syrup for use in making sweets. How much of 49% sugar syrup should she add to 4 liters of 9% sugar syrup to obtain a mixture that contains 45% sugar?

43. A landscaper wants to use a mixture of soil and sand to ensure proper water drainage. He has just 47 cubic meters of soil that is 32% sand and an unlimited quantity of soil that is 38% sand. How many cubic meters of the 38% soil should he add to the 32% to create a mixture that is 35% sand?

FINDING THE MISSING NUMBER

To find a missing number when you're given the average, **use the sum**. If the average of four numbers is 7, then the sum of those four numbers is 4×7 , or 28. Suppose that three of the numbers are 3, 5, and 8. These numbers add up to 16 of that 28, which leaves 12 for the fourth number.

Example: Martin's average score after four tests is 89. What score on the fifth test would bring Martin's average up to exactly 90?

Solution: The best way to deal with changing averages is to go by the way of the sums. Use the old average to figure out the total of the first 4 scores: Sum of first 4 scores = $4 \times 89 = 356$. And use the new average to figure out the total he needs after the fifth score. Sum of 5 scores = $4 \times 90 = 450$. To get his sum up from 456 to 450, Martin needs to score $450 - 356 = 94$

Sample Questions:

44. In Kara's math class, all tests count equally. So far, Kara has taken 4 of the 5 tests in math class this marking period and earned scores of 88%, 95%, 86% and 79% respectively. What is the minimum score Kara needs on the third test to have a test average of 85%?

45. Andy has grades of 84, 65, and 76 on three math tests. What grade must he obtain on the next test to have an average of exactly 80 for the four tests?

46. Linda computed the average of her six biology test scores by mistakenly adding the totals of five scores and dividing by five, giving her an average score of 88. When Linda realized her error, she recalculated and included the sixth test score of 82. What is the average of Linda's six biology tests?

AVERAGE OF EVENLY SPACED NUMBERS

When you have a set of evenly spaced numbers, such as 10 consecutive numbers, or all the odd numbers between 5 and 35, you can use a shortcut rather than the averages formula. To find the average of evenly spaced numbers, just **average the smallest and the largest**. The average of all the integers from 13 through 77 is the same as the average of 13 and 77.

$$\frac{13+77}{2} = \frac{90}{2} = 45.$$

Sample Questions:

47. What is the average of all integers from 15 through 99?

48. What is the difference between the average of all consecutive numbers from 33 through 99 and the average of all even numbers from 2 through 24?

49. What is the average of the set {3, 5, 7, 9, 11, 13, 15, 17, 19, 21}?

REPEATING DECIMAL

To find a particular digit in a repeating decimal, note the **number of digits in the cluster that repeats**. If there are two digits in that cluster, then every second digit is the same. If there are three digits in that cluster, then every third digit is the same. And so on. For example, the decimal equivalent of $1/27$ is $.037037037\dots$ which is best written $\overline{0.037}$.

There are three digits in the repeating cluster, so every third digit is the same: 7. To find the 50th digit, look for the multiple of 3 just less than 50—that's 48. The 48th digit is 7, and with the 49th digit the pattern repeats with 0. So, the 50th digit is 3.

Sample Questions:

50. In the number 0.1666, what digit is coming next?

51. In the number, 0.191919, what digit is coming next?

52. What is the 217^{th} digit after the decimal point in the repeating decimal $0.\overline{3456}$?

53. When $4/11$ is written as a decimal, what is the 100^{th} digit after the decimal point?

OTHER REPEATING NUMBERS

This idea of repeating can be used in other number examples.

Example: If the first day of the year is a Monday, what is the 260^{th} day?

Solution: Sketch out a little calendar until you see a pattern: Day 1 is Monday, 2 is Tuesday, 3 is Wednesday, 4, Thursday, 5, Friday, 6 Saturday, 7 Sunday, 8 Monday, and so on. Notice that Sundays are always multiples of 7. Pick a multiple of 7 close to 260, such as 259. That means Day 269 is a Sunday, so Day 260 is one more so a Monday.

Sample Questions:

54. March 7th is a Wednesday, what day of the week is September 7th?

55. If Valentine's Day is on a Friday (in a non-Leap Year), what day of the week is Christmas?

SOLVING PROBLEMS USING THE SLOT METHOD

Combination problems ask you how many different ways a number of things could be chosen or combined.

The rules for combination problems on the ACT are straightforward.

- Figure out the number of slots you need to fill.
- Fill in those slots.
- Find the product.

On a more difficult problem, you may run into a combination with more restricted elements. Just be sure to read the problem carefully before attempting it. If the question makes your head spin, leave it and return to it later, or pick your Letter of the Day and move on. For good measure, though, here's what one of those tougher ones might look like.

Example: At the school cafeteria, 2 boys and 4 girls are forming a lunch line. If the boys must stand in the first and last places in line, how many different lines can be formed?

Solution: These restrictions might make this problem seem daunting, but this is where the slot method is really helpful. We have six spots in line to fill, so draw six slots:

— — — — — —

Fill in the restricted spots first. The problem tells us that the two boys must stand in the first and last places in line. This means that either of the boys could stand in first place, and then the other boy will stand in last. This means that we have two options for the first place, but only one for the last.

2 1

Now do the same with the unrestricted parts. Any of the four girls could stand in the second spot.

2 4 1

Now, since one of the girls is standing in the second spot, there are only three left to stand in the third spot, and so on, and so on.

2 4 3 2 1 1

Now, as ever, just take the product, to find that the correct answer is choice (C), 48.

2 × 4 × 3 × 2 × 1 × 1 = 48

Sample Questions:

56. In the word HAWKS, how many ways is it possible to rearrange the letters if none repeat and the letter W must go last?

57. A classroom has 10 tables that will seat up to 4 students each. If 20 students are seated at tables, and NO tables are empty, what is the greatest possible number of tables that could be filled with students?

58. How many different positive three-digit integers can be formed if the three digits 3, 4, and 5 must be used in each of the integers?

MULTI STEP SLOT METHOD AND PROBABILITY PROBLEMS

Some problems are more complicated and require combining several problem solving methods into one problem.

Example: Elias has to select one shirt, one pair of pants, and one pair of shoes. If he selects at random from his 8 shirts, 4 pairs of pants, and 3 pairs of shoes, and all his shirts, pants, and shoes, and all are different colors, what is the likelihood that he will select his red shirt, black pants, and brown shoes?

- a. $1/3$
- b. $1/4$
- c. $1/15$
- d. $1/32$
- e. $1/96$

Solution:

We have three slots to fill here, and we want to find the product of the three.

$$\begin{array}{cccccc} \underline{8} & \times & \underline{4} & \times & \underline{3} & = 96 \\ \text{Shirts} & & \text{Pants} & & \text{Shoes} & \end{array}$$

Of the 96 possible arrangements, an ensemble of red shirt, black pants, and brown shoes is only one.

Therefore, we can go to the $\frac{\text{part}}{\text{whole}}$ ratio to find $\frac{\text{part}}{\text{whole}} = \frac{\text{red, black, brown}}{\text{All arrangements}} = \frac{1}{96}$, choice (E).

Example: If 3 people all shake hands with each other, there are a total of 3 handshakes. If 4 people all shake hands with each other, there are a total of 6 handshakes. How many total handshakes will there be if 5 people all shake hands with each other?

Solution:

A B C D E

Now you only need to identify the number of different combination.

AB, AC, AD, AE, BC, BD, BE, CD, CE, DE

There are 10 distinct combinations, meaning 10 total handshakes.

Sample Question:

59. An integer from 100 through 999, inclusive, is to be chosen at random. What is the probability that the number chosen will have 0 as at least 1 digit?

60. As part of a probability experiment, Elliott is to answer 4 multiple-choice questions. For each question, there are 3 possible answers, only 1 of which is correct. If Elliott randomly and independently answers each question, what is the probability that he will answer the 4 questions correctly?

61. For every positive 2-digit number, x , with tens digit t and units digit u , let y be the 2-digit number formed by reversing the digits of x . Which of the following expressions is equivalent to $x - y$?

62. In a geometric sequence, the quotient of any two consecutive terms is the same. If the third term of a geometric sequence is 8 and the fourth term is 16, then what is the second term?

Answer Key

1. 175
2. 450.8
3. 2,500
4. 236
5. \$25.43
6. \$36.56
7. $n\$25.50 / 6 \text{ shirts} = \4.25 per shirt $\$18.00 / 4 \text{ shirts} = \4.50 per shirt $\$21.00 / 5 \text{ shirts} = \4.20 per shirt - this is the best buy
8. $360 \text{ pages} / 12 \text{ hours} = 30 \text{ pages per hour}$ $400 \text{ pages} / 30 \text{ pages per hour} = 13 \frac{1}{3} \text{ hours} = 13 \text{ hours and 20 minutes.}$
9. $11/15$
10. $5x/3$
11. \$6.86
12. 84.5
13. 16
14. $a/2$
15. 90%
16. 46%
17. Need to have 3.2 or greater, so B+ to be able to play
18. 0.1 hr or 6 min longer
19. 10 hrs
20. 4.12 hours
21. 3.21 hours
22. 120
23. 280
24. $11/13$
25. $4/9$
26. $5/8$
27. 68%
28. \$22.29
29. $2/9$
30. $2/5$
31. There are presently 1,620 men at the college.
32. 90 additional women
33. $15:4$
34. There are 52 more girls than boys in the school.
35. 50 mph
36. $7 + \frac{7}{3}d$
37. 6 weeks
38. 0.8 seconds
39. 48 mph
40. 4.2 miles
41. To obtain the desired product, 10 gallons of the oil containing 59% eugenol should be added.
42. To obtain a mixture that contains 45% sugar, 36 liters of the 49% sugar syrup should be added.
43. The landscaper should add 47 cubic meters of the soil that is 38% sand.
44. 77%
45. 95%
46. 87%
47. 42
48. 22
49. 9
50. 6
51. 1
52. 3
53. 6
54. Friday
55. Thursday
56. 24
57. 3
58. 6
59. $171/900$
60. $1/81$
61. $9(x - y)$
62. 4