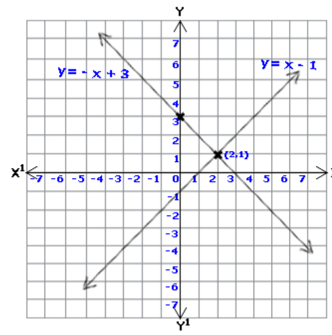
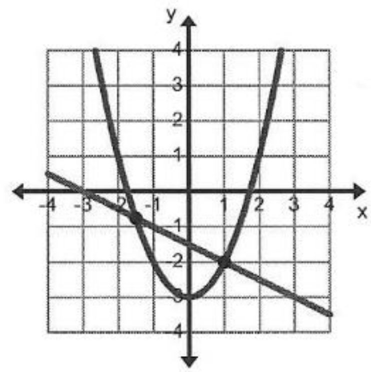
Name:___________________________________ Date:_________________________

Math 3 Coordinate Geometry Part 2
Graphing Solutions

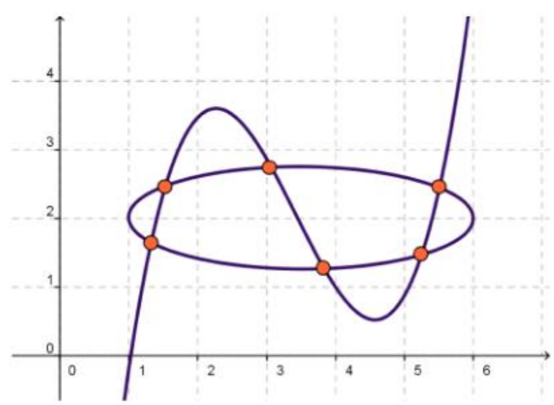
SOLVING SYSTEMS OF EQUATIONS GRAPHICALLY
The solution of two linear equations is the point where the two lines intersect. For example, in the graph at right we see that the solution for the equations y = -x + 3 and y = x -1 is the point (2, 1). It is the place where the answers are the same for both equations.



Likewise, the solution of two polynomials are the points of intersection. For example to find when $f(x) = g(x)$ if $f(x) = x^2 - 3$ and $g(x) = \frac{-1}{2}x + 1$ we look at the graphs and see where they intersect. The two points are (1,-2) and approximately (-1.5, -0.75)



Sample Questions:
1. What are the solutions for the polynomial of degree 3 and the ellipse as shown in the following graph? Round answers to the nearest 0.25.

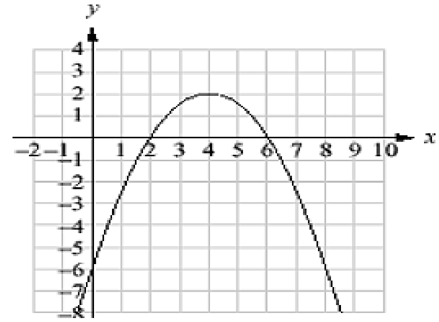


KEY FEATURES OF GRAPHS:
MINIMUMS AND MAXIMUMS

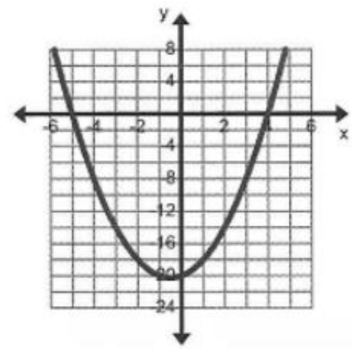
A minimum is the lowest point on a graph and a maximum is the highest point on a graph. In a parabola the minimum or maximum will be the y value of the vertex.

Sample Questions:

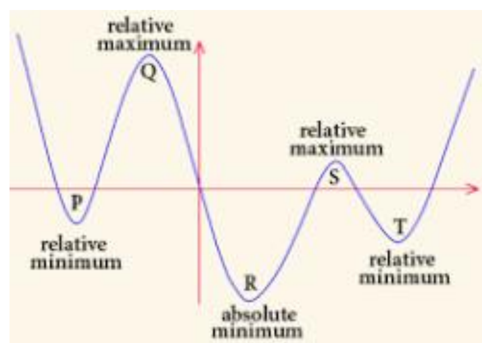
2. What is the maximum of the parabola in the following graph?



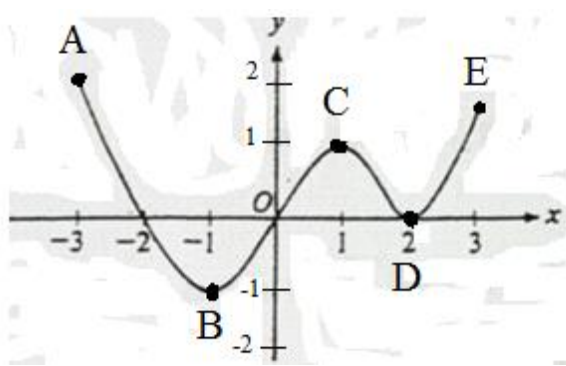
3. What is the minimum of the parabola in the following graph?



A graph may have more than one minimum and/or maximum. A "relative maximum" is a bump in the graph that may not be the highest point on the graph, but is higher than the points around it. Likewise a "relative minimum" is a dip that may not be the lowest point on the graph, but is lower than the points around it.



4. In the following graph describe each point as an absolute maximum, relative maximum, absolute minimum, or relative minimum



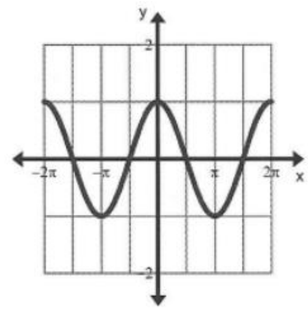
DOMAIN AND RANGE

In mathematics the domain of a function is the set of "input" values for which the function is defined. In other words, it is all the x values of the function. In the graph for question 4 the domain is [-3, 3]. All the x values are between -3 and 3. The range is the set of "output" or "y" values for which the function is defined. In the graph for question 4 the range is [-1, 2]. All the y values are between -1 and 2. When writing the interval for domain and range, we use different brackets to indicate whether the endpoints are included. [-3, 3] means $-3 \le x \le 3$ and (-3, 3) means $-3 < x < 3$ and [-3,3) means $-3 \le x < 3$.

Sample Questions:

5. Using bracket notation, write the domain interval for $-1 < x < 6$.

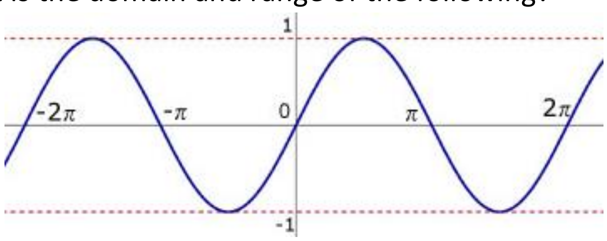
6. Using bracket notation, write the range interval for $0 < y \le 3$.



For the function $f(x) = \cos(x)$ the domain is $(-\infty, \infty)$ and the range is [-1, 1]. Note that whenever ∞ or $-\infty$ are in the domain or range we use the ( ) brackets next to the infinity symbol.

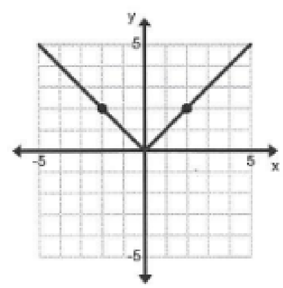
Sample Questions:

What is the domain and range of the following?



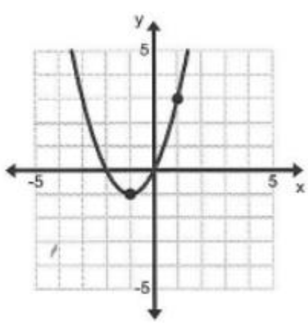
A. f(x) = sin(x)

7.



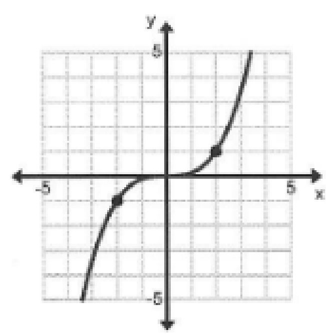
B. f(x) = |x|

8.



C. f(x) = $x^2 + 2x$

9.



D. f(x) = $\frac{1}{8}x^3$

10.

ASYMPTOTES

An asymptote is a line that a graph approaches, as x increases or decreases, but does not intersect.

VERTICAL ASYMPTOTES AND DOMAIN

Vertical asymptotes occur when the function is a fraction and the denominator is equal to zero, but the numerator is not zero. If both the numerator and denominator are equal to zero then the point is undefined, but it does not create an asymptote. Either way, this situation affects the domain. Remember that domain is the set of all x values that are "defined." This means that when a point is "undefined" it cannot be in the domain. Therefore the question "What is the domain of this function?" usually means to search for any possible places that are undefined.

Example: Find the domain of $f(x) = \frac{(x-2)}{(x+5)}$. X cannot be -5, but it can be anything else so the domain is $(-\infty, -5) \cup (-5, \infty)$ or sometimes they just say x $\neq -5$

To find vertical asymptotes of an equation, factor the numerator and denominator and look for zeros in the denominator.

$f(x) = \frac{2(x-3)(x+7)}{(x-1)(x+1)}$ Here if x = 1 or if x = -1 the denominator would be zero.

Question: can x = 3 or -7? Yes. Getting a 0 in the ***numerator*** is fine, but getting a 0 in the ***denominator*** is not.

Sample Questions:

Find the domain for questions 12-14.

11. $f(x) = \frac{3(x-1)(x+8)}{(x-3)(x+2)}$

12. $f(x) = \frac{(x+6)(x+5)}{(x-4)(x-1)}$

13. $f(x) = \frac{(x-1)(x+2)}{x(x-2)}$

FACTORING EQUATIONS

Finding the zeros, range, or asymptote is easy when the equation is factored like we saw in the last three questions. This is one of the reasons why we learn how to do this, because this is so important in graphing we are going to practice.

Example: Factor the quadratic equation $x^2 - 9x + 20 = 0$

Solution: We're trying to get it to look something like this ( )( ). When we multiply what is inside the first parenthesis with what is in the second parenthesis it will turn back into $x^2 - 9x + 20$. This is basically doing "FOIL" or the distributive property backwards. We need to find two numbers which add up to 9 and multiply to make 20. These numbers are 4 and 5. $(x-4)(x-5) = 0$. You can check these answers by multiplying it out and checking to see if it goes back to the original equation.
$x^2 - 9x + 20 = 0$. You can also check by substituting 4 and 5 into the equation and checking to see if the answer really is 0. $(4)^2 - 9(4) + 20 = 16 - 36 + 20 = 0$. Yes. $(5)^2 - 9(5) + 20 = 25 - 45 + 20 = 0$. Yes.

Example: Factor the equation $n^2 + 4n - 12$.

Solution: We need two numbers that multiply to make 12 and subtract to make 4. 6 and 2 work. (Note that we add the two numbers is the last term is positive and subtract the two numbers is the last term is negative). $(n+6)(n-2)$. We can verify this by multiplying it out and seeing if we get $n^2 + 4n - 12$. We can also check by substituting -6 and 2 into the original equation and checking to see if the answer really is 0.

Sample Questions:
Factor each function completely.

14. $b^2 + 8b + 7$

15. $n^2 - n - 56$

16. $n^2 - 5n + 6$

17. $x^2 - 11x + 10$

DIFFERENCE BETWEEN TWO SQUARES

When factoring, it is very importance to be able to recognize the form called "the difference between two squares." It looks like this $(a + b)(a - b) = a^2 - b^2$ in generic form.

Example: Factor the quadratic equation $x^2 - 16$.
Solution: This is the same as $(x)^2 - (4)^2$. it is the difference between two squares so the answer is $(x + 4)(x - 4)$

Example: Factor the quadratic equation $2x^2 - 18$.
Solution: The first step is to factor out a 2 and rewrite the equation as $2(x^2 - 9)$. Then we need to recognize that the remaining part in the parenthesis are a difference between two squares. $2([x]^2 - [3]^2)$. So the answer is $2(x + 3)(x - 3)$

Example: Factor the quadratic equation $q^4 - p^4$.
Solution: This is the same as $(q^2)^2 - (p^2)^2$ which is the difference between two squares. So we simplify to $(q^2 + p^2)(q^2 - p^2)$ from here we need to recognize that one of the factors is the difference of two squares and can be simplified further. So the answer is $(q^2 + p^2)(q + p)(q - p)$.
Factor each quadratic completely:

Sample Questions:
18. $m^2 - 1$

19. $n^2 - 100$

20. $2x^2 - 50$

21. $3r^2 - 3$

22. $x^2 - y^2$

23. $2x^4 - 8z^4$

24. $mx^2 - 9m$

Use the function $f(x) = \frac{x^2 + 8x+7}{x^2-4}$ for questions 26-31.

25. Factor the denominator completely $x^2 - 4$.

26. What is the domain of the function?

27. Does the function have any asymptotes?

28. If so where are they?

29. Factor the numerator $x^2 + 8x + 7$ completely.

30. What are the zeros of the function (where does the graph cross the x-axis)?

Use the function $f(x) = \frac{x^2-25}{x^2-10x+9}$ for questions 32-37.

31. Factor the denominator completely

32. What is the domain of the function?

33. Does the function have any asymptotes?

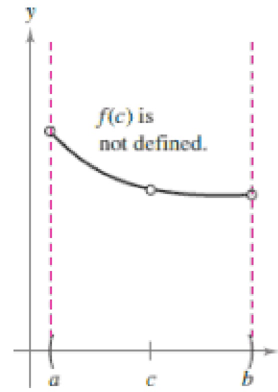
34. If so where are they?

35. Factor the numerator completely.

36. What are the zeros of the function (where does the graph cross the x-axis)?

UNDEFINED POINT

Sometimes there might be a zero in the numerator and the denominator at the same time. This doesn't create an asymptote, instead it creates a point on the graph that is undefined. On a graph we draw a little open circle to represent the place where the graph is undefined.



Example: $f(x) = \frac{(x+1)(x-3)}{(x-2)(x-3)}$

In the numerator we see that if x = -1 or if x = 3 the numerator should equal 0. In the denominator we see that if x = 2 or if x = 3 then the function would be undefined. Since the point at x =3 would be a zero in BOTH the numerator and the denominator it creates an undefined point rather than an asymptote.

Sample Questions:

Use the function $f(x) = \frac{x^2+8x+7}{x^2-1}$ for questions 38-45.

37. Factor the numerator $x^2 + 8x + 7$ completely.

38. What are the zeros of the function (where does the graph cross the x-axis)?

39. Factor the denominator completely $x^2 - 1$.

40. What is the domain of the function?

41. Does the function have any asymptotes?

42. If so where are they?

43. Does the function have one or more undefined points?

44. If so where are they?

Use the function $f(x) = \frac{x^2-25}{x^2-4x-5}$ for questions 46-52.

45. Factor the denominator completely

46. Factor the numerator completely.

47. What is the domain of the function?

48. Does the function have any asymptotes or undefined points?

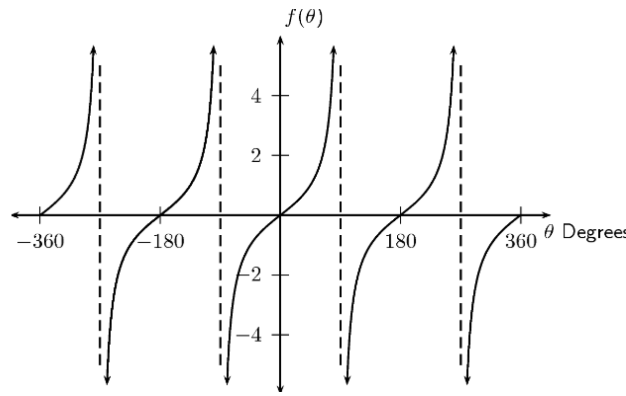
49. If so where are the asymptotes?

50. If so where are the undefined points?

51. What are the zeros of the function (where does the graph cross the x-axis)?

COMPARING FUNCTIONS OF DIFFERENT FORMS

It is important to be able to make the connection between the information on the graph and the information in an equation. When we answer a question we want to understand what we're finding and what it means. In the graph for tangent I can tell from the graph that the function is a fraction (ie rational) because there are vertical asymptotes. I can tell that the numerator is 0 at -360°, -180°, 0°, 180°, and 360° because the graph crosses the x-axis at these points. I can also tell that the denominator is 0 at -270°, -90°, 90°, and 270° because there are vertical asymptotes at those points.

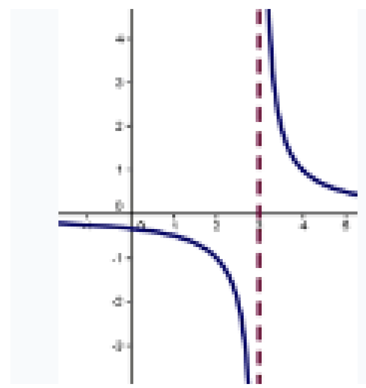


Sample Questions:

52. Could the graph at right represent the equation $f(x) = \frac{(x-1)(x+2)}{x(x-2)}$?
53. List 2 reasons why or why not?

54. Could the graph at right represent the equation $f(x) = x^2 - 2x - 3$?
55. List 2 reasons why or why not?

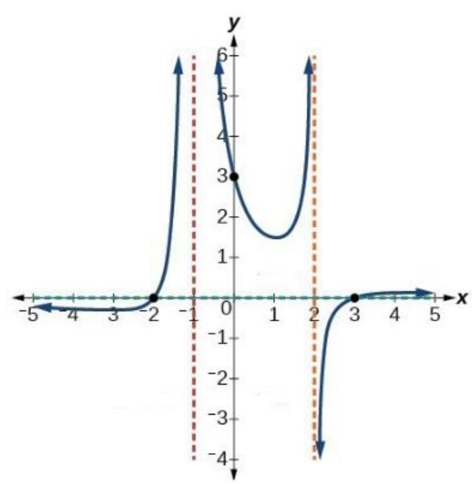
56. Could the graph at right represent the equation $f(x) = \frac{1}{(x-3)}$?
57. List 2 reasons why or why not?



58. Do you think the function for the graph at right is a fraction (i.e. rational)?
59. Why or why not?

60. Are there any places where the numerator is 0?
61. If so where are they?

62. Are there any places where the denominator is 0?
63. If so where are they?



REVIEWING POWER FUNCTION GRAPHS AND END BEHAVIOR

End behavior refers to the directions that the left end and right end of the function are headed. By noticing patterns, we can generalize the end behavior of any polynomial. For any polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2\ a_1x \ + a_0$ we only need to look at the first term $a_n x^n$. We're looking for two things: first, is the exponent odd or even? Second, is the coefficient positive (> 0) or negative (< 0)? If the exponent is even then the ends will be like those of a parabola, an upturned smile if the coefficient is positive $(\text{towards} + \infty)$. Contrariwise the edges will be like a downturned frown if the coefficient is negative $(\text{towards} - \infty)$. Likewise, if the exponent is odd then one end will turn upward and the other downward. If the coefficient is positive then the left end points down $(\text{towards} - \infty)$ and the right end points up $(\text{towards} + \infty)$. If the coefficient is negative then left moves toward +∞ and the right moves toward $- \infty$.

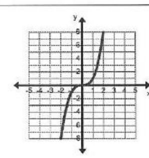
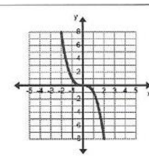
Refer to the polynomial $f(x) = -4x^3 + 6x^2 - 5x + 7$ for questions 65-67.

Sample Questions:

64. Is the exponent of the first term odd or even?

65. Is the coefficient of the first term positive or negative?

66. Describe the left and right end behavior for the polynomial

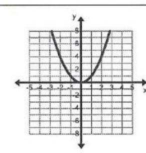
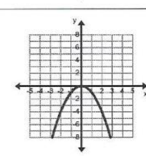
Function	Graph	Left End Behavior	Right End Behavior
$f(x)=x^3$		$\lim_{x\to-\infty} f(x) = ?$	$\lim_{x\to\infty} f(x) = ?$
$f(x)=-x^3$		$\lim_{x\to-\infty} f(x) = ?$	$\lim_{x\to\infty} f(x) = ?$

Refer to the polynomial $f(x) = 3x^2 + 7x + 6$ for questions 68-70

67. Is the exponent of the first term odd or even?

68. Is the coefficient of the first term positive or negative?

69. Describe the left and right end behavior for the polynomial

Function	Graph	Left End Behavior	Right End Behavior
$f(x)=x^2$		$\lim_{x\to-\infty} f(x) = +\infty$	$\lim_{x\to\infty} f(x) = +\infty$
$f(x)=-x^2$		$\lim_{x\to-\infty} f(x) = -\infty$	$\lim_{x\to\infty} f(x) = -\infty$

GRAPHING FUNCTIONS AND RATIONAL FUNCTIONS
Use key features to graph various kinds of functions

Example: Graph the function that has the following key features:

Left end behavior → $-\infty$
Right end behavior → ∞
Crosses the x-axis at 1
Touches the x-axis at -3
Relative minimum at (-1,-7)
Relative maximum at (-3, 0)
Domain [-5, 2]
Range [-8, 6]

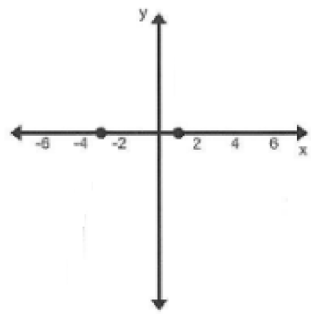


Figure 1

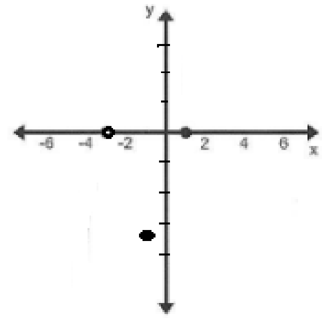


Figure 2

Solution:
Plot the zeros on the graph as shown in figure 1. Plot the minimum and maximum as shown in figure 2. Note the end behavior as shown in figure 3 making sure that it fits within the domain and range of the function. Connect all the points as shown in figure 4.

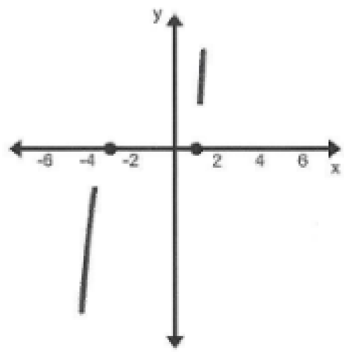


Figure 3

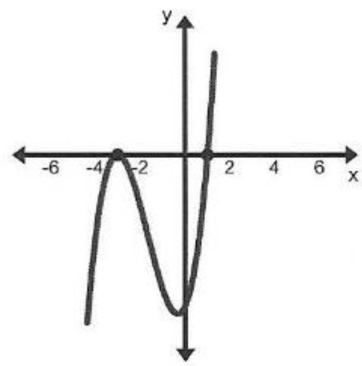
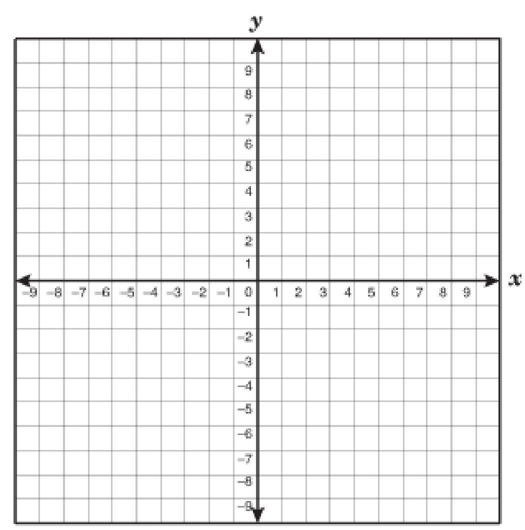


Figure 4

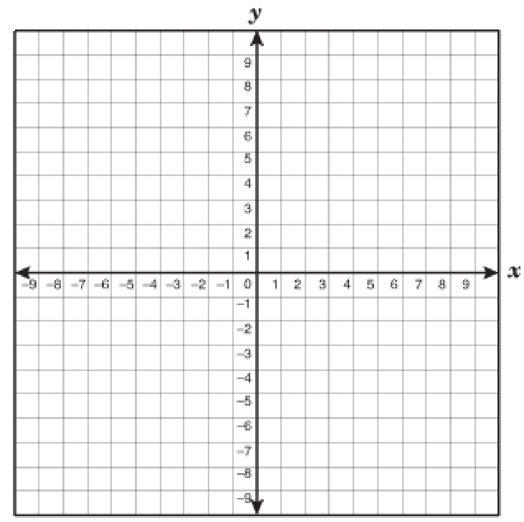
70. Graph the function that has the following key features:

Left end behavior → $-\infty$
Right end behavior → ∞
Crosses the x-axis at -2, 5, 9
Relative minimum at (7, -1)
Relative maximum at (1, 3)

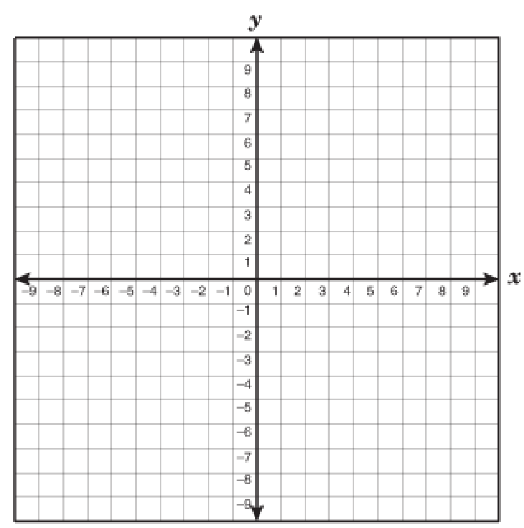


The next step is to be able to recognize key features from a function to be able to graph the function.

71. Graph the function $f(x) = x^2 + 4x$
(hint: What is the end behavior? What are the zeros? What is the vertex?)

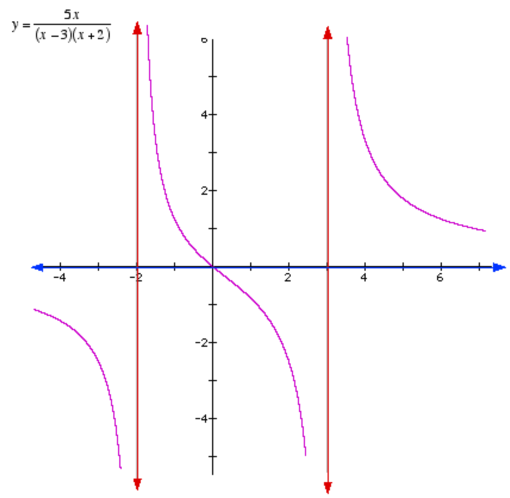


72. Graph the function $f(x) = x^2 - 6x + 8$
(hint: What is the end behavior? What are the zeros? What is the vertex?)



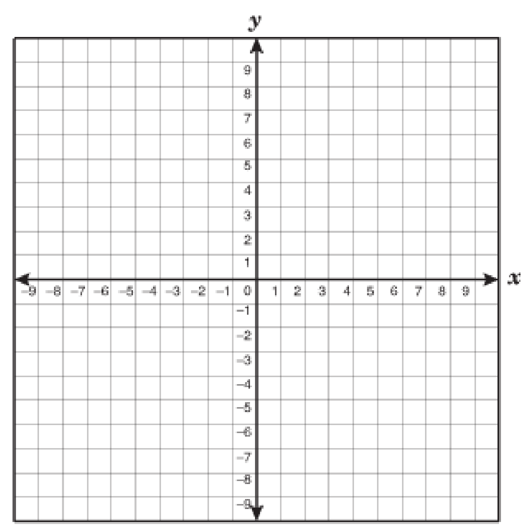
Example of graphing rational function:

Graph the function $f(x) = \frac{5x}{x^2-x-6}$



The first step is to factor the numerator (if necessary) and the denominator (if necessary). In this function we don't need to factor the numerator, but we do need to factor the denominator. Rewriting the function we get: $f(x) = \frac{5x}{(x-3)(x+2)}$ From the denominator we can tell that there are two asymptotes, one at x = 3 and another at x = -2. That means the graph never crosses those lines. From the numerator we can tell that the graph crosses the x-axis at x = 0. Then we test a few values near the asymptotes to see how the function behaves. Does it go up or down?

73. Graph the function $f(x) = \frac{2x-2}{x^2-4}$.



(hint: Is this a rational function (ie a fraction)? How does that affect the graph? Factor the numerator and the denominator if necessary. What are the zeros? What is the domain? Are there any asymptotes? Test a few numbers near the asymptotes to see how the function behaves.

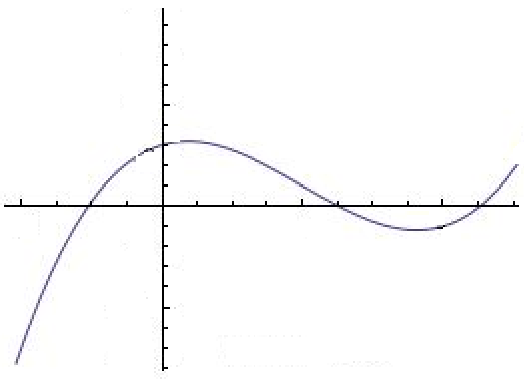
Answers

1. (1.25, 1.75), (1.5, 2.5), (3, 2.75), (3.75, 1.25), (5.25, 1.5), (5.5, 2.5)
2. 2
3. -20
4. A. absolute max; B. absolute minimum; C. relative maximum; D. relative minimum; E. relative maximum
5. (-1, 6)
6. (0, 3]
7. domain $(-\infty,\infty)$; range [-1, 1]
8. domain (-∞,∞); range [0, ∞)
9. domain (-∞,∞); range [-1, ∞)
10. domain (-∞,∞); range (-∞,∞)
11. x≠-2, 3 or (-∞,2) ∪ (-2,3) ∪ (3, ∞)
12. x≠1,4 or(-∞,1) ∪ (1,4) ∪ (4, ∞)
13. x≠0,2 or (-∞,0) ∪ (0,2) ∪ (2, ∞)
14. (b +1)(b+7)
15. (n-8)(n+7)
16. (n-2)(n-3)
17. (x-1)(x-10)
18. (m+1)(m-1)
19. (n-10)(n+10)
20. 2(x+5)(x-5)
21. 3(r+1)(r-1)
22. (x+y)(x-y)
23. $2(x^2 + 2z^2)(x^2 - 2z^2)$
24. m(x+3)(x-3)
25. (x+2)(x-2)
26. x≠-2, 2 or (-∞,-2) ∪ (0,2) ∪ (2, ∞)
27. yes
28. at x = -2 and x=2
29. (x+1)(x+7)
30. at x = -1 and x = -7
31. (x-1)(x-9)
32. x≠1,9 or (-∞,1) ∪ (1,9) ∪ (9, ∞)
33. yes
34. at x=1 and x=9
35. (x+5)(x-5)
36. at x = -5 and x = 5
37. (x+1)(x+7)
38. at x = -7
39. (x+1)(x-1)
40. x ≠-1, 1 or (-∞,-1) ∪ (-1,1) ∪ (1, ∞)
41. yes
42. at x = 1
43. yes
44. at x = -1
45. (x-5)(x+1)
46. (x+5)(x-5)
47. x ≠ -1, 5 or (-∞,-1) ∪ (-1,5) ∪ (5, ∞)
48. yes
49. x = 1
50. x = 5
51. x = 5
52. no
53. the function has 2 asymptotes, one at x=0 and another at x=2 but the graph only has one at x=3; the function has 2 zeros at x = 1 and at x = -2 but the graph doesn't cross the x-axis at all.
54. no
55. the function is not a rational function (fraction) but the graph has an asymptote so it much be a rational function; the function has 2 zeros, one at x=3 and one at x= -1 but the graph doesn't cross the x-axis at all.
56. yes
57. the function is a rational function (fraction) which matches the graph; the function has an asymptote at x=3 just like the graph; the function doesn't have any zeros and the graph doesn't cross the x-axis so that matches
58. yes
59. it has asymptotes
60. yes
61. x= -2, x= 3
62. yes
63. x= -1, x= 2
64. odd
65. negative
66. left end → ∞; right end → $-\infty$
67. even
68. positive
69. left end → ∞; right end → ∞

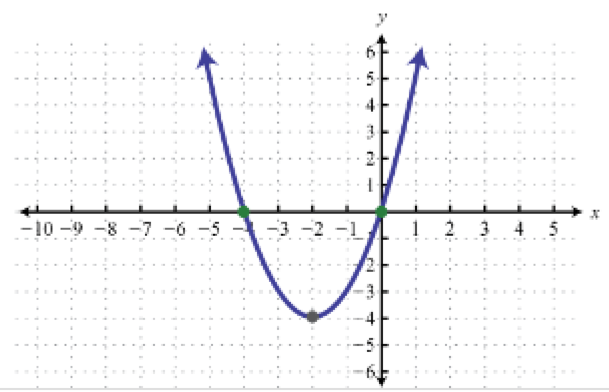
70. see graph A
Left end behavior → $-\infty$
Right end behavior → ∞
Crosses the x-axis at -2, 5, 9
Relative minimum at (7, -1)
Relative maximum at (1, 3)
Domain [-4, 10]
Range [-8, 10]

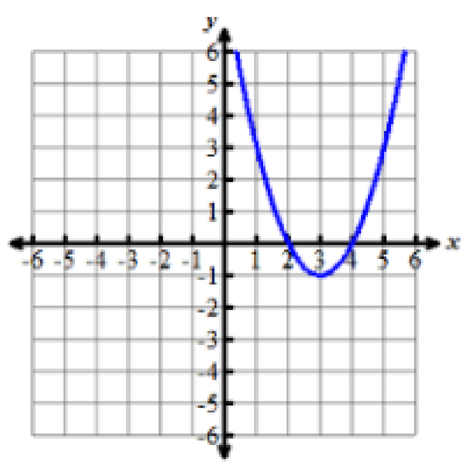
71. see graph B
72. see graph C
73. see graph

Graph A

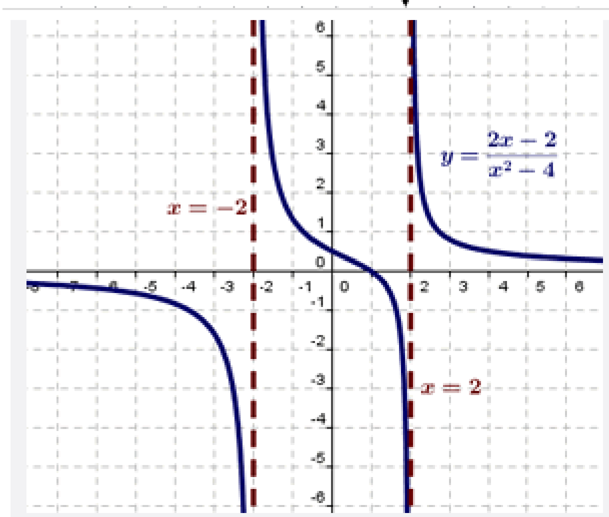


Graph B





Graph C



Graph D