Math 2 Variable Manipulation Part 8
Forms and Uses of Exponential Functions

MONEY AND INTEREST

An exponential function is a function of the form $f(x) = ab^x$ where a and b are constants and $a \neq 0$, $b > 0$, and $b \neq 1$. Money and interest are calculated using various exponential function formulas.

Simplified Interest: Exponential functions can also be of the form $A = P(1 + r)^t$. This is the simplified interest formula. Each part of the formula has a specific meaning. The principal, P, is the original amount of money that is deposited. The interest rate, r, is expressed as a decimal and represents the growth rate of the investment. Time, t, is the number of years that the money remains in the account. The amount, A, after t years can be calculated by using the formula.

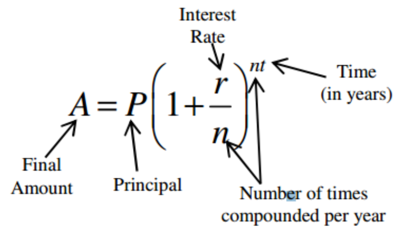
Example: Austin deposits $450 into a savings account with a 2.5% interest rate. How much money will be in the account after 5 years?
Solution: Substitute the known values into the equation for $A = P(1 + r)^t$ and solve
$A = 450(1 + 0.025)^5$
$450(1.025)^5 = 509.13$
Austin will have $509.13 in his account after 5 years.

Example: Emily invested $1250 after 2 years she had $1281.45. What was the interest rate, if the interested was assessed once a year?
Solution: Substitute the known values into the equation $A = P(1 + r)^t$ and solve for unknown
$1281.45 = 1250(1 + r)^2$
$\frac{1281.45}{1250} = (1 + r)^2$ Take square root of both sides
$\sqrt{\frac{1281.45}{1250}} = \sqrt{(1\ +\ r)^2}$
$\pm 1.0125 = 1 + r$
Subtract 1 from both sides and solve for the positive and negative to get two separate answers
$r = 0.0125$ or $r = -2.0125$
The interest rate is positive; therefore it is 0.0125 or 1.25%.

There are times when the value of an item decreases by a fixed percent each year. This can be modeled by the formula $A = P(1 - r)^t$ where P is the initial value of the item, r is the rate at which its value decreases, and A is the value of the item after t years.

Example: Jeff bought a new car for $27,500. The car's value decreases by 8% each year. How much will the car be worth in 15 years?
Solution: Substitute the known values into the equation for simplified interest $A = P(1 - r)^t$
$A = 27{,}500\ (1 + 0.08)^{15} = 7873.18$
Jeff should sell the car for at least $7873.18.

Compound Interest: Interest is commonly assessed multiple times throughout the year. This is referred to as compound interest because the interest is compounded or applied more than once during the year. The formula is A = P $(1 + \frac{r}{n})^{nt}$ where n is the number of times that the interest is compounded during the year.



Example: Cyndi received a lot of money for her high school graduation and she plans to invest $1000 in a Dream CD. A CD is a certified account that pays a fixed interest rate for a specified length of time. Cyndi chose to do a 3 year CD with a 0.896% interest rate compounded monthly. How much money will she have in 3 years?

Solution: Substitute known values into the equation A = P $(1 + \frac{r}{n})^{nt}$ and solve

$A = 1000\,(1 + \frac{0.00896}{12})^{12*3} = 1000\,(1 + \frac{0.00896}{12})^{36} = 1027.23$

Cyndi will have $1027.23 after 3 years.

Example: Sam invested some money in a CD with an interest rate of 1.15% that was compounded quarterly. How much money did Sam invest if he had $1500 after 10 years?

Solution: Substitute known values into the equation A = P $(1 + \frac{r}{n})^{nt}$ and solve for the unknown

$1500 = P\,(1 + \frac{0.015}{4})^{4*10}$

$1500 = P\,(1 + 0.00375)^{40}$

$\frac{1500}{(1+0.00375)^{40}} = P$

P = 1291.42

Ten years ago Sam invested $1291.42.

Continuous Interest: There is another interest formula where the interest is assessed continuously. The formula is $A = Pe^{rt}$. It uses the same P, r, and t from the other interest formulas but it also utilizes the Euler constant e. Similar to pi, e is an irrational number. It is defined to be e = 2.718281828. It is also called the natural base. e^x can be found on most calculators, so just plug in to calculate.

Example: Eva invested $750 in a savings account with an interest rate of 1.2% that is compounded continuously. How much money will be in the account after 7 years?

Solution: Substitute known values into the equation $A = Pe^{rt}$ and solve

$A = 750e^{0.012*7} = 750e^{0.084} = 815.72$

Eva will have $815.72 after 7 years.

Sample Questions:

1. Bobbi is investing $10,000 in a money market account that pays 5.5% interest quarterly. How much money will she have after 5 years?

2. Joshua put $5,000 in a special savings account for 10 years. The account had an interest rate of 6.5% compounded continuously. How much money does he have?

3. Analeigh is given the option of investing $12,000 for 3 years at 7% compounded monthly or at 6.85% compounded continuously. Which option should she choose and why?

4. Mallory purchased a new Road Glide Ultra motorcycle for $22,879. Its value depreciates 15% each year. How much could she sell it for 8 years later?

5. Jace invested $12,000 in a 3-year Dream CD with interest compounded annually. At the end of the 3 years, his CD is worth $12,450. What was the interest rate for the CD?

6. Jaron has a savings account containing $5,000 with interest compounded annually. Two years ago, it held $4,500. What was the interest rate?

7. Lindsey needs to have $10,000 for the first semester of college. How much does she have to invest in an account that carries an 8.5% interest rate compounded monthly in order to reach her goal in 4 years?

EXPONENTIAL GROWTH AND DECAY

Money is not the only real world phenomenon that can be modeled with an exponential function. Other phenomena such as populations, bacteria, radioactive substances, electricity, and temperatures can be modeled by exponential functions.

Exponential Growth: A quantity that grows by a fixed percent at regular intervals is said to have exponential growth. The formula for uninhibited growth is $A(t) = A_0e^{kt}$, where A_0 is the original amount, t is the time and k is the growth constant. This formula is similar to the continuous interest formula $A = Pe^{rt}$. Both formulas are continuously growing or growing without any constraints.

$$A\left(t\right) = A_0 e^{kt}$$

Time

Growth Constant

Final amount

Initial Amount

Example: A colony of bacteria that grows according to the law of uninhibited growth is modeled by the function

$A(t) = 100e^{0.045t}$, where A is measured in grams and t is measured in days.

(a) Determine the initial amount of bacteria.

(b) What is the growth constant of the bacteria?

(c) What is the population after 7 days?

Solution:

(a) A_0 is the initial amount and in the equation $A_0 = 100$, therefore the initial amount is 100 grams.

(b) k is the growth constant and in the equation k = 0.045 , therefore the growth rate is 0.045/day.

(c) Substitute 7 in for time, t, then evaluate.

$A(7) = 100e^{0.045(7)} = 137.026$ After 7 days, there will be 137.026 grams of bacteria.

Exponential Decay: A quantity that decreases by a fixed percent at regular intervals is said to have exponential decay. The formula for uninhibited decay is $A(t) = A_0e^{kt}$, k = < where A_0 is the original amount, t is the time and k is the constant rate of decay.

The decay formula is the same as the growth formula. The only difference is that when $k > 0$ the amount increases over time making the function have exponential growth and when $k < 0$ the amount decreases over time making the function have exponential decay.

Example: A dinosaur skeleton was found in Vernal, Utah. Scientists can use the equation $A(t) = 1100e^{-0.000124t}$, where A is measured in kilograms and t is measured in years, to determine the amount of carbon remaining in the dinosaur. This in turn helps to determine the age of the dinosaur bones.

(a) Determine the initial amount of carbon in the dinosaur bones.

(b) What is the growth constant of the carbon?

(c) How much carbon is left after 5,600 years?

Solution:

(a) A_0 is the initial amount and in the equation $A_0 = 1{,}100$, therefore the initial amount is 1,100 kilograms.

(b) k is the growth constant and in the equation $k = -0.000124$. Since k is negative, therefore the carbon is decaying at a rate 0.000124/year.

(c) Substitute 5600 in for time, t, then evaluate.

$A(5600) = 1100e^{-0.000124(5600)} = 549.311$ After 5600 years, there will be 549.311 kilograms of carbon remaining.

Exponential Growth and Decay with Growth Factor: Exponential growth and decay can also be a function of the form

$$A(t) = A_0 b^t = A_0(1 + r)^t$$

where A_0 is the initial amount, $b = 1 + r$ is the growth factor, and $A_0 \neq 0$, $b > 0$, and $b \neq 1$.

If $r < 0$, is exponential decay   If $r > 0$, is exponential growth

Example: A culture of bacteria obeys the law of uninhibited growth. Initially there were 500 bacteria present. After 1 hour there are 800 bacteria.

a. Identify the growth rate.

b. Write an equation to model the growth of the bacteria

c. How many bacteria will be present after 5 hours?

Solution:

a. Find the common ratio. $\frac{a_{n+1}}{a_n} = \frac{800}{500} = 1.6$ Rewrite b in the form of 1+ r. b = 1.6 = 1 + 0.6

Identify the growth rate. r = 0.6 = 60%   The bacterial is increasing at a rate of 60% each hour.

b. Since $A(t) = A_0 b^t = 500(1.6)^t$

c. Substitute t = 5 into the equation and evaluate.

$A(t) = 500(1.6)^5 = 5242.88$ There are 5,242 bacteria present after 5 hours.

Example: Michael bought a new laptop for $1,800 last year. A month after he purchased it, the price dropped to $1,665.

a. Identify the growth rate.

b. Write an equation to model the value of the computer.

c. What will the value of the computer be after 9 months?

Solution:

a. Find the common ratio. $\frac{a_{n+1}}{a_n} = \frac{1665}{1800} = 0.925$ Rewrite b in the form of 1+ r. b = 0.0925 = 1 – 0.075

Identify the growth rate. r = -0.075 = -7.50%   The value of the computer is decreasing at a rate of 7.5% each month.

b. Since $A(t) = A_0 b^t = 1800(0.925)^t$

c. Substitute t = 9 into the equation and evaluate.

$A(t) = 1800(0.925)^t = 892.38$ After 9 months, the computer is worth $892.38.

Example: The population of West Jordan was 106,863 in 2011. The population is growing at a rate of 5% each year.

a. Write an equation to model the population growth.

b. If this trend continues, what will the population be in 2020?

Solution:

a. $A_0 = 106863$, r = 5%, 1 + r = 1 + 0.05 = 1.05

$A(t) = A_0(1 + r)^t = 106863(1.05)^t$

b. t = 2020 - 2011 = 9

$A(t) = 106863(1.05)^9 = 165779.587$ After 9 years, the population of West Jordan is 165,779 people.

Example: A culture of 200 bacteria is put in a petri dish and the culture doubles every hour.

a. Write an equation to model the bacteria growth.

b. If this trend continues, how many bacteria will there be in 5 hours?

Solution:

a. $A_0 = 200$, r = 100%, 1 + r = 1 + 1 = 2

$A(t) = A_0(1 + r)^t = 200(2)^t$

b. t = 5

$A(t) = 200(2)^5 = 6400$ After 5 hours, there are 6,400 bacteria.

Sample Questions:

8. India is one of the fastest growing countries in the world. $A(t) = 574e^{0.026t}$ describes the population of India in millions t years after 1974.
    a. What was the population in 1974?
    b. Find the growth constant.
    c. What will the population be in 2030?

9. The amount of carbon-14 in an artifact can be modeled by $A(t) = 16e^{-0.000121t}$, where A is measured in grams and t is measured in years.
    a. How many grams of carbon-14 were present initially?
    b. Find the growth constant.
    c. How many grams of carbon-14 will be present after 5,715 years?

10. A bird species is in danger of extinction. Last year there were 1,400 birds and today only 1,308 of the birds are alive.
    a. Identify the growth rate.
    b. Write an equation to model the population.
    c. If this trend continues, what will the population be in 10 years?

11. There is a fruit fly in your house. Fruit fly populations triple every day until the food source runs out.
    a. Write an equation to model the fruit fly growth.
    b. If this trend continues, how many fruit flies will there be at the end of 1 week?

12. In 2003 the population of Nigeria was 124,009,000. It has a growth rate of 3.1%.
    a. Write an equation to model the population growth since 2003.
    b. If this trend continues, what will the population be in 2050?

13. Utah's population was 2,763,885 in 2010 and 2,817,222 in 2011, find the growth rate.

Answer Key

1. \$13,140.67
2. \$9,577.70
3. Analeigh should choose the compounded monthly because after 3 years it is \$14,795.11 while the continuous is \$14,737.67.
4. \$6,234.31
5. 1.23%
6. 5.4%
7. \$7,126.24
8. a. 574,000,000 people b. 0.026 people/year c. 2,461,754,033 people
9. 8g
10. a. decrease 6.57% b. $B(t) = 1400(0.934)^t$ c. 707 or 708 birds
11. 16,384 fruit flies
12. a. $P(t) = 124{,}009{,}000(1.031)^t$ b. 497,511,091 people
13. 1.93%