Math 2 Variable Manipulation Part 6
System of Equations

SYSTEM OF EQUATIONS INTRODUCTION
A "system" of equations is a set or collection of equations that you deal with all together at once. Linear equations (ones that graph as straight lines) are simpler than non-linear equations, and the simplest linear system is one with two equations and two variables.

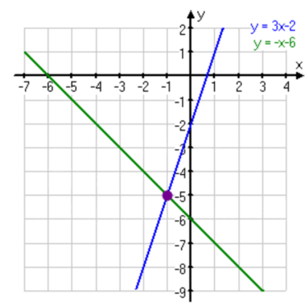
Consider the linear equation $y = 3x - 5$. A "solution" to this equation was any x, y-point that "worked" in the equation. So (2, 1) was a solution because, plugging in 2 for x:

$3x - 5 = 3(2) - 5 = 6 - 5 = 1 = y$

On the other hand, (1, 2) was not a solution, because, plugging in 1 for x:

$3x - 5 = 3(1) - 5 = 3 - 5 = -2$

...which did not equal y (which was 2, for this point). Of course, in practical terms, picking random points, plugging them in, and checking to see if they "work" in the equation is not an efficient way to solve the equation. Instead, select an x-value and then calculate the corresponding y-value.



Now consider the following two-variable system of linear equations:

$y = 3x - 2$
$y = -x - 6$

Since the two equations above are in a system, we deal with them together at the same time. In particular, we can graph them together on the same axis system.

The purple point is a solution to the system, because it lies on both of the lines. In particular, this purple point marks the intersection of the two lines. Since this point is on both lines, it thus solves both equations, so it solves the entire system of equation.

This relationship is always true: For systems of equations, "solutions" are "intersections". You can confirm the solution by plugging it into the system of equations, and confirming that the solution works in each equation.

Example: Determine whether either of the points (–1, –5) and (0, –2) is a solution to the given system of equations.

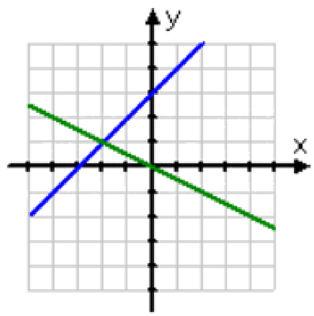
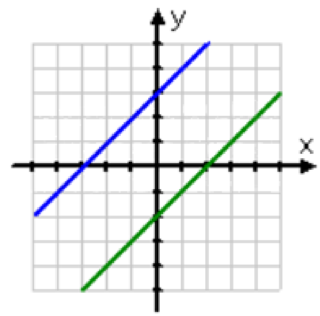
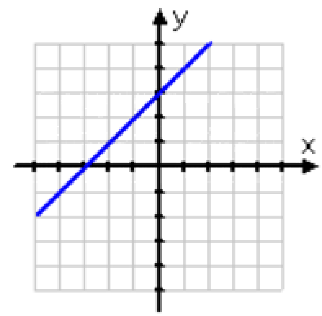
$y = 3x - 2$
$y = -x - 6$

Solution: To check the given possible solutions, plug the x- and y-coordinates into the equations, and check to see if they work. First, Substitute (–1, –5) into the equations: -5 = 3(-1) -2 = -5 (true) and -5 = -(-1) – 6 = -5 (also true). Since the given point works in each equation, it is a solution to the system. Next, Substitute (0, –2) into the equations: -2 = 3(0) -2 = -2 (true) and -2 = -(0) – 6 = -6 (not true). The solution works in one of the equations. But to solve the system, it has to work in both equations, so this "solution" does not work. The answer is that only the point (–1, –5) is a solution to the system

Note: You can solve for two variables only if you have two distinct equations. If you have one variable, you only need one equation, but if you have two variables, you need two distinct equations. Two forms of the same equation will not be adequate. If you have three variables, you need three distinct equations, and so on.

SOLVING A SYSTEM OF EQUATIONS BY GRAPHING

When graphically finding intersections of lines. For two-variable systems, there are then three possible types of solutions:

Independent system: one solution point	Case 2	Case 3
		

The first graph above, "Case 1", shows two distinct non-parallel lines that cross at exactly one point. This is called an "independent" system of equations, and the solution is always some *x*,*y*-point.

The second graph above, "Case 2", shows two distinct lines that are parallel. Since parallel lines never cross, then there can be no intersection; that is, for a system of equations that graphs as parallel lines, there can be no solution. This is called an "inconsistent" system of equations, and it has no solution.

The third graph above, "Case 3", appears to show only one line. Actually, it's the same line drawn twice. These "two" lines, really being the same line, "intersect" at every point along their length. This is called a "dependent" system, and the "solution" is the whole line.

This shows that a system of equations may have one solution (a specific *x*,*y*-point), no solution at all, or an infinite solution (being all the solutions to the equation). You will never have a system with two or three solutions; it will always be one, none, or infinitely-many.

Solving systems of equations "by graphing," is a bit tricky because the only way you can find the solution from the graph is *IF* you draw a very neat axis system, *IF* you draw very neat lines, *IF* the solution happens to be a point with nice neat whole-number coordinates, and *IF* the lines are not close to being parallel.

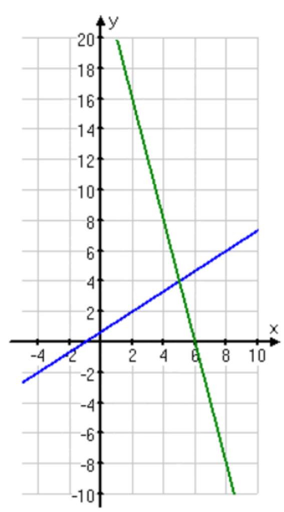
Example: Solve the following system by graphing

$2x - 3y = -2$

$4x + y = 24$

Solution: Guess values of x and solve for values of y
Then graph to find intersection

x	$y = (2/3)x + (2/3)$	$y = -4x + 24$
–4	–8/3 + 2/3 = –6/3 = –2	16 + 24 = 40
–1	–2/3 + 2/3 = 0	4 + 24 = 28
2	4/3 + 2/3 = 6/3 = 2	–8 + 24 = 16
5	10/3 + 2/3 = 12/3 = 4	–20 + 24 = 4
8	16/3 + 2/3 = 18/3 = 6	–32 + 24 = –8



Solution is (5, 4)

SOLVING A SYSTEM OF EQUATIONS BY SUBSTITUTION

The method of solving "by substitution" works by solving one of the equations (you choose which one) for one of the variables (you choose which one), and then plugging this back into the other equation, "substituting" for the chosen variable (so that you only have one equation with one variable) and solving for the other. Then you back-solve for the first variable.

Example: Solve the following system by substitution.

$2x - 3y = -2$
$4x + y = 24$

Solution: The idea here is to solve one of the equations for one of the variables, and plug this into the other equation. It does not matter which equation or which variable you pick. There is no right or wrong choice; the answer will be the same, regardless. But — some choices may be better than others.

For instance, in this case, it would probably be simplest to solve the second equation for "y =", since there is already a y by itself in the equation. Solving the first equation for either variable, would yield fractions, and solving the second equation for x would also give fractions. It wouldn't be "wrong" to make a different choice, but it would probably be more difficult. So, solve the second equation for y:

$4x + y = 24$
$y = -4x + 24$

Now plug this in ("substitute it") for "y" in the first equation, and solve for x:

$2x - 3(-4x + 24) = -2$
$2x + 12x - 72 = -2$
$14x = 70$
$x = 5$

Now, plug this x-value back into either equation, and solve for y. But since there is already an expression for "y =", it will be simplest to just plug into this:

$y = -4(5) + 24 = -20 + 24 = 4$

Then the solution is $(x, y) = (5, 4)$.

Warning: If I had substituted "$-4x + 24$" expression into the same equation used to solve for "y =", the answer is a true, but useless, statement:

$4x + (-4x + 24) = 24$
$4x - 4x + 24 = 24$
$24 = 24$

Twenty-four does equal twenty-four, but who cares? So when using substitution, make sure to substitute into the *other* equation, or you'll just be wasting your time.

Note: If you correctly solve a system of equations and get something obviously wrong line -12 = 3, then the answer is “No Solution”. This means that the lines do not intersect—there is no solution to the system of equations. If you correctly solve a system of equations and get something like 4 = 4, you know that both equations are actually the same equation—the same line, they are not independent from each other, so every point on the line is a solution.

Sample Questions:
Solve each system by substitution:

1. $3x - 5y = 17$
   $y = -7$

2. $-3x - 3y = 3$
   $y = -5x - 17$

3. $-7x - 2y = -13$
   $x - 2y = 11$

4. $-3x + 3y = 4$
   $-x + y = 3$

5. $-3x - 4y = 2$
   $3x + 3y = -3$

6. $-7x + 2y = 18$
   $6x + 6y = 0$

SOLVING A SYSTEM OF EQUATIONS BY ELIMINATION

The method of elimination for solving systems of equations is also called the addition method. This method is similar to the method you probably learned for solving simple equations. If you had the equation "$x + 6 = 11$", you would write "–6" under either side of the equation, and then you'd "add down" to get "$x = 5$" as the solution.

$$\begin{array}{rcl} x + 6 &=& 11 \\ \underline{\quad -6} && \underline{-6} \\ x &=& 5 \end{array}$$

You'll do something similar with the elimination or addition method so that one of the variables will cancel out and you only have one equation with one variable which is easily solvable and then you can substitute the answer into one of the equations and solve for the second variable.

Example: Solve the following system using the elimination method.

$$2x + y = 9$$
$$3x - y = 16$$

Solution: Note that, when the equations are added, the y's will cancel out. So draw an "equals" bar under the system, and add down:

$$\begin{array}{l} 2x + y = 9 \\ \underline{3x - y = 16} \\ 5x \quad\ = 25 \end{array}$$

Now I can divide through to solve for $x = 5$, and then back-solve, using either of the original equations, to find the value of y. The first equation has smaller numbers, so I'll back-solve in that one:

$$2(5) + y = 9$$
$$10 + y = 9$$
$$y = -1$$ Then the solution is $(x, y) = (5, -1)$.

It doesn't matter which equation you use for the back solving; you'll get the same answer either way. If I'd used the second equation, I'd have gotten:

$$3(5) - y = 16$$
$$15 - y = 16$$
$$-y = 1$$
$y = -1$...which is the same result as before.

Some equations are not set up as easily as the previous example. You may need to modify one or both of the equations to make them into two equations that are easily added to cancel out one of the variables. One option is to multiply one equation by negative 1 or another number that will when added to the other equation will cancel out one of the variables so that you only have one equation with one variable. The main strategy is to combine the equations in such a way that one of the variables cancels out.

Example: Solve the following system using elimination method

$$4x + 3y = 8$$
$$x + y = 3$$

Solution: Multiply both sides of the second equation by -3 to get: $-3x - 3y = -9$. Now add the equations; the $3y$ and the $-3y$ cancel out, leaving $x = -1$:

$$\begin{array}{l} \quad 4x + \cancel{3y} = 8 \\ \underline{+(-3x - \cancel{3y} = -9)} \\ \quad\ x \qquad = -1 \end{array}$$

Plug into one of the original equations and you'll find that $y = 4$.

Then the solution is $(x, y) = (-1, 4)$

Example: Solve the following system using elimination method

$4x - 3y = 25$
$-3x + 8y = 10$

Solution: Since nothing easily cancels, there is not a number than you can multiply one equation by to cancel, both equations must be multiplied to create a cancellation. One option is to multiply to convert the *x*-terms to $12x$'s or the *y*-terms to $24y$'s. Since the 12 is smaller than 24, multiply the first equation by 3 and the second equation by 4 to cancel the *x*-terms. (You would get the same answer in the end if you set up the *y*-terms to cancel. They are both the "right way"; it was just a choice. You could make a different choice, and that would be just as correct.)

$$\begin{array}{rcr} 4x - 3y = 25 & \xrightarrow{3R_1} & 12x - 9y = 75 \\ -3x + 8y = 10 & \xrightarrow{4R_2} & \underline{-12x + 32y = 40} \\ & & 23y = 115 \end{array}$$

Solving, I get that $y = 5$. Using the first equation for back-solving:

$4x - 3(5) = 25$
$4x - 15 = 25$
$4x = 40$
$x = 10$

The solution is $(x, y) = (10, 5)$

Sample Questions:

Solve each system by elimination:

7. $x - y = 11$
   $2x + y = 19$

8. $-6x + 5y = 1$
   $6x + 4y = -10$

9. $7x + 2y = 24$
   $8x + 2y = 30$

10. $-7x + y = -19$
$-2x + 3y = -19$

11. $16x - 10y = 10$
$-8x - 6y = 6$

12. $-x - 7y = 14$
$-4x - 14y = 28$

13. $-4y - 11x = 36$
$20 = -10x - 10y$

14. $-9 + 5y = 4x$
$11x = -20 + 9y$

15. $-16y = 22 + 6x$
$-11y - 4x = 15$

MORE SYSTEM OF EQUATIONS
There are many types of systems of equations in many varied forms. You can solve for two variables only if you have two distinct equations. If you have one variable, you only need one equation, but if you have two variables, you need two distinct equations. Two forms of the same equation will not be adequate. If you have three variables, you need three distinct equations, and so on. **Use logic to combine the equations in such a way that one of the variables cancels out**.

Example: If $mn = k$ and $k = x^2n$, and $nk \neq 0$, m is equal to?
Solution: Since $mn = k$ AND $x^2n = k$, set the values of k equal to each other.
$mn = x^2n$ then, solve for m which gives $m = x^2$

Example: A system of linear equations is shown below.
$4y - 2x = 8$
$4y + 2x = 8$
Which describes the graph of the system of linear equations in the standard (x,y) coordinate plane?

a. A single line with positive slope
b. A single line with negative slope
c. Two distinct intersecting lines
d. Two parallel lines with positive slope
e. Two parallel lines with negative slope

Solution: Since this problem is asking for the graph, solving by graphing is the easiest way to proceed.
Reorganize each problem into the point-slope form
$y = \frac{1}{2}x + 2$
$y = -\frac{1}{2}x + 2$
Either draw the two lines on a graph or recognize from the equations that both lines intersect at point (0, 2) and one has a negative slope and the other a positive slope. Choices a and b cannot be correct because there are two distinct lines. Neither can d and e because the slopes are not the same. C is correct.

Example: If $(a + b)^2 = 25$ and $(a - b)^2 = 45$, then $a^2 + b^2 = ?$
Solution: Because both of the equations are squares, write them out and then add together
$(a + b)^2 = 25$ becomes $a^2 + 2ab + b^2 = 25$
$(a - b)^2 = 45$ becomes $\underline{a^2 - 2ab + b^2 = 45}$ Then add these two equations together
$\frac{2a^2 + 2b^2}{2} = \frac{70}{2}$ Divide both sides by 2
Answer is: $a^2 + b^2 = 35$

Example: For what value of n would the following system of equations have an infinite number of solutions?

$$3a + 6b = 12$$
$$2a + 4b = 2n$$

Solution: For this system of equations to have an infinite number of solutions, they have to be the same equation of the line. Therefore, if you add them together, the variables will be eliminated and the constants will be equivalent. One way to compare these equations to see if they are the same line is to try to modify them so both variables will be eliminated. Multiply the first equation by 2 and the second equation by -3.
$2(3a + 6b = 12) = 6a + 12b = 24$
$-3(2a + 4b = 2n) = \underline{-6a - 12b = -6n}$
$0 = 24 - 6n$ Solving for $n = 4$

Sample Questions:

16. When $x/y = 4$, $x^2 - 12y^2$?

17. If $x + y = 13$ and $2y = 16$, what is the value of x?

18. If $x + 4y = 5$ and $5x + 6y = 7$, then $3x + 5y = ?$

19. The equations below are linear equations of a system where a, b, and c are positive integers.

$$ay + bx = c$$
$$ay - bx = c$$

Which of the following describes the graph of at least 1 such system of equations in the standard (x,y) coordinate plane?

I. 2 parallel lines
II. 2 intersecting lines
III. A single line

a. I only
b. II only
c. III only
d. I or II only
e. I, II, or III

20. If $x = 6a + 3$ and $y = 9 + a$, which of the following expresses y in terms of x?

a. $y = x + 51$
b. $y = 7x + 12$
c. $y = 9 + x$
d. $y = \frac{57 + x}{6}$
e. $y = \frac{51 + x}{6}$

21. Which of the following equations expresses z in terms of x for all real numbers x, y, and z, such that $x^5 = y$ and $y^3 = z$?

a. $z = x$
b. $z = 3/5\ x$
c. $z = 3x^5$
d. $z = x^8$
e. $z = x^{15}$

SYSTEM OF EQUATIONS WORD PROBLEMS

There are many word problems that include system of equations. Translate the words into equations and use various system of equations methods to solve. Sometimes there may be three variables, but you can combine in such a way to cancel out two at a time and still get a solution to the problem.

Example: The larger of two numbers exceeds twice the smaller number by 9. The sum of twice the larger and 5 times the smaller number is 74. If a is the smaller number, which equation below determines the correct value of a?

a. 5(2a + 9) + 2a = 74
b. 5(2a - 9) + 2a = 74
c. (4a + 9) + 5a = 74
d. 2(2a + 9) + 5a = 74
e. 2(2a - 9) + 5a = 74

Solution: Using L for the larger number and a for the smaller number (the last sentence states that they want the smaller number to be represented by the letter a), the first sentence translates to L = 2a + 9. The second sentence translates to 2L + 5a = 74. Substituting the first equation into the L in the second equation yields 2(2a + 9) + 5a = 74 which is D.

If the problem asked for a solution, first distribute the 2 to get

4a + 18 + 5a = 74 then combine like terms and subtract 18 from both sides

9a = 56 then divide by 9 to get

a = 56/9

Example: The average of a and b is 6 and the average of a, b, and c is 11. What is the value of c?

Solution: This problem looks like there are three variables and only two equations, but if you follow through, you will find out that the answer is obtainable. First, translate the English sentence into a math equation. Remember that average is the sum of the numbers divided by the total numbers.

$\frac{a+b}{2} = 6$

$\frac{a+b+c}{3} = 11$

Solve the first equation for a + b

a + b = 2(6) = 12

Then plug this answer for a + b into the second equation and solve for c

$\frac{12+c}{3} = 11$

12 + c = 33

-12 -12

c = 21

Sample Questions:

22. The difference of two number is 3 and their sum is 13. Find the numbers.

23. The sum of the real numbers a and b is 13. Their difference is 5. What is the value of ab?

24. The sum of the digits of a certain two-digit number is 7. Reversing its digits increases the number by 9. What is the number?

25. Two small pitchers and one large pitcher can hold 8 cups of water. One large pitcher minus one small pitcher constitutes 2 cups of water. How many cups of water can each pitcher hold?

26. Margie is responsible for buying a week's supply of food and medication for the dogs and cats at a local shelter. The food and medication for each dog costs twice as much as those supplies for a cat. She needs to feed 164 cats and 24 dogs. Her budget is $4240. How much can Margie spend on each dog for food and medication?

27. The school that Stefan goes to is selling tickets to a choral performance. On the first day of ticket sales the school sold 3 senior citizen tickets and 1 child ticket for a total of $38. The school took in $52 on the second day by selling 3 senior citizen tickets and 2 child tickets. Find the price of a senior citizen ticket and the price of a child ticket.

28. A boat traveled 210 miles downstream and back. The trip downstream took 10 hours. The trip back took 70 hours. What is the speed of the boat in still water? What is the speed of the current?

29. The senior classes at High School A and High School B planned separate trips to New York City. The senior class at High School A rented and filled 1 van and 6 buses with 372 students. High School B rented and filled 4 vans and 12 buses with 780 students. Each van and each bus carried the same number of students. How many students can a van carry? How many students can a bus carry?

COMPOSITE FUNCTIONS

Composite functions are similar to system of equations in that there are two functions to combine into one function to solve. However, the procedure is a little different. If the problems is to find $f(g(x)$, replace all values of x in $f(x)$ with the value of $g(x)$ then simplify or solve for x. Notice that $f(g(x)$ is different from $g(f(x)$ which is figured out by the same procedure, but you replace all the values of x in $g(x)$ with the value of $f(x)$ then simplify.

Example: $f(x) = -4x + 9$ and $g(x) = 2x – 7$, find $f(g(x))$.
Solution: Substitute $g(x)$ which is 2x - 7 into every x in $f(x)$ which is -4x + 9

$f(g(x)) = -4(2x – 7) + 9$
$= -8x + 28 + 7$
$= -8x + 37$

The answer is: $f(g(x)) = -8x + 37$

Example: $f(x) = -4x + 9$ and $g(x) = 2x – 7$, find $g(f(x))$.
Solution: Notice that this is the same two equations, but they are asking for $g(f(x))$ instead of $f(g(x))$. This time substitute $f(x)$ which is $-4x + 9$ into every x in $g(x)$ which is $2x – 7$

$g(f(x)) = 2(-4x + 9) – 7$
$= -8x + 18 – 7$
$= -8x + 11$

The answer is: $g(f(x)) = -8x + 11$

Note: It is important to notice what the problem is asking so you can substitute correctly.

Example: If $h(x) = 3x – 5$ and $g(x) = 2x^2 – 7x$, find $g(h(x))$
Solution: Substitute $3x – 5$ into all the x values in g(x)

$2(3x - 5)^2 - 7(3x – 5)$
$2(9x^2 - 30x + 25) - 7(3x\ 5)$
$18x^2 - 60x + 50 - 21x + 35$
$18x^2 - 81x + 85$

The answer is: $g(h(x)) = 18x^2 - 81x + 85$

Functions may also be represented by data tables. If this is the case, look up the data on the tables to get the answers.

Example: For the 2 functions f(x) and g(x), tables of values are shown below. What is the value of $g(f(-1))$?

x	$f(x)$
−3	−6
−1	2
1	−3
3	9

x	$g(x)$
1	0
2	3
3	8
4	15

Solution: Instead of having two equations and substituting $f(x)$ into $g(x)$, and simplifying, just look up the values on the chart for $f(-1)$ on the $f(x)$ chart and compare that answer to the chart for $g(x)$.
The value for x in the $f(x)$ chart for $f(-1)$ is 2. The value for g(x) when x = 2 in the g(x) chart is 3.
So the answer for $g(f(-1)) = 3$

Sample Questions:

30. Given $f(x) = 3x + 5$ and $g(x) = x^2 - x + 7$, which of the following is an expression for $f(g(x))$?
    a. $3x^2 - 3x + 26$
    b. $3x^2 - 3x + 12$
    c. $x^2 - x + 12$
    d. $9x^2 + 25x + 27$

31. Consider the functions $f(x) = \sqrt{x}$ and $g(x) = 7x + b$. In the standard (x,y) coordinate plane, $y = f(g(x))$ passes through (4,6). What is the value of b?

32. Given $f(x) = 4x + 1$ and $g(x) = x^2 - 2$, which of the following is an expression for $f(g(x))$?

a. $-x^2 + 4x + 1$
b. $x^2 + 4x - 1$
c. $4x^2 - 7$
d. $4x^2 - 1$
e. $16x^2 + 8x - 1$

33. If $f(x) = x^2 - 4x + 2$ and $g(x) = 3x - 7$, find $f(g(x))$.

34. If $g(x) = -6x + 5$ and $h(x) = -9x - 11$, find $g(h(x))$.

35. If $f(x) = \sqrt{2x - 5}$ and $g(x) = 5x^2 - 3$, find $g(f(x))$.

Answers

1. $(-6, -7)$
2. $(-4, 3)$
3. $(3, -4)$
4. No Solution
5. $(-2, 1)$
6. $(-2, 2)$
7. $(10, -1)$
8. $(-1, -1)$
9. $(6, -9)$
10. $(2, -5)$
11. $(0, -1)$
12. $(0, -2)$
13. $(-4, 2)$
14. $(-1, 1)$
15. $(-1, -1)$
16. $4y^2$
17. 5
18. 6
19. II only (B)
20. $y = \frac{51+x}{6}$ (E)
21. $z = x^{15}$ (E)
22. 5 and 8
23. 36
24. 34
25. 2 and 4
26. $40 for each dog
27. senior citizen ticket: $8, child ticket: $14
28. boat: 12 mph, current: 9 mph
29. Van: 18, Bus: 59
30. $3x^2 - 3x + 26$ (a)
31. 8
32. $4x^2 - 7$ (c)
33. $9x^2 - 54x + 79$
34. $54x + 71$
35. 10X - 28