Name:_____ Date:_____

Math 2 Proportion & Probability Part 2
Percent, Ratio, Proportions, Rate
Average, Patterns, Sequences

COMBINED PERCENT INCREASE AND DECREASE

When there are multiple percent increases and/or decreases, and the question asks for the combined percent increase or decrease, the easiest and most effective strategy is to pick 100 for the original value and see what happens.

Example: A price went up 10% one year, and the new price went up 20% the next year. What was the combined percent increase?

Solution: First year: 100 + (10% of 100) = 110. Second year: 110 + (20% of 110) = 132.

That's a combined 32% increase.

Sample Questions:

1. On Friday, a computer was priced at $800. On the following Wednesday, the price was reduced by 15%. On the following Friday, the price was further reduced by 20%. What percent of the original price was the final price?

2. A number is increased by 25% and the resulting number is then decreased by 20%. The final number is what percent of the original number

3. A shoe store charges $39 for a certain type of sneaker. This price is 40% more than the amount it costs the shoe store to buy one pair of these sneakers. At an end-of-the-year sale, sales associates can purchase any remaining sneakers at 20% off the shoe store's cost. How much would it cost an employee to purchase a pair of sneakers of this type during the sale (excluding sales tax)?

PART-TO-PART AND PART-TO-WHOLE RATIOS

A part-to-part ratio can be turned into two part-to-whole ratios by putting **each number in the original ratio over the sum of the parts.** If the ratio of males to females is 1 to 2, then the males-to-people ratio is 1 to 3 and the females-to-people ratio is 2 to 3.

Sample Questions:

4. Esther is making 2 ¼ gallons of punch for a large party. While mixing the punch, she uses ½ gallon of pineapple juice. What fraction of the punch consists of pineapple juice?

5. In a poll, 44 people were in favor of constructing a new high school, 58 were against it, and 8 people had no opinion. What fraction of those people polled were in favor of constructing a new high school?

MULTI STEP RATIO PROBLEMS

The most common ratio problems involve a comparison between two quantities. These ratios are called two-term ratios. There are three basic steps you must do first, when working any ratio problems: Change the quantities to the same units; then reduce the ratio to its simplest form.

Example: What is the ratio of 6 minutes to 8 hours?

Solution: First, change the hours to minutes: 8 hours = 8 × 60 = 480 minutes

Write the ratio as a fraction and simplify:

$\frac{6}{480} = \frac{1}{80}$ The ratio of 6 minutes to 8 hours is 1:80.

Remember: Make sure that there are the same items in the numerator and denominator.

Example: A room is 16 feet, 8 inches long, and the ratio of the length to the width is 4 to 5. What is the width of the room?

Solution: Step 1: Since the length is given in both feet and inches, let's convert it to inches using the fact that 1 foot equals 12 inches. To find how many inches are in 16 feet, we multiply 16 feet by 12 inches:

16 feet, 8 inches = (16 × 12) + 8 = 192 + 8 = 200

We found that the length is 200 inches.

Step 2: Let x represent the width. We can now set up the equation:

$\frac{\text{width}}{\text{length}} = \frac{4}{5} = \frac{x}{200}$; The width is 160 inches.

Let's now convert inches to feet so that the units for the width are consistent with the units for the length. Since 1 foot is 12 inches, we divide 160 inches by 12 to find out how many feet are in 160 inches:

160:12 = 13, with the remainder of 4 inches. The width is 13 feet, 4 inches. The room width is 13 feet, 4 inches.

Sample Questions:

6. At a small college, the ratio of men to women is 9:4. If there are presently 720 women, how many men are at the college?

7. How many additional women would it take to reduce the ratio of men to women to 2:1?

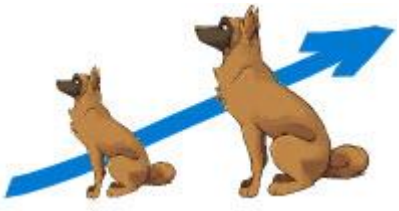
8. The ratio of x to z is 3 to 5, and the ratio of y to z is 1 to 5. What is the ratio of x to y?

9. If x:y = 5:2 and y:z = 3:2, what is the ratio of x:z ?

10. A school has 300 students. If the ratio of boys to girls is 31 to 44, how many more girls are there in the school?

DIRECTLY PROPORTIONAL AND INVERSELY PROPORTIONAL

Directly proportional



Directly proportional means as one amount increases, another amount increases at the same rate. The symbol for "directly proportional" is $\propto$ (Don't confuse it with the symbol for infinity)

Example: You are paid $20 an hour. How much you earn is directly proportional to how many hours you work. Work more hours, get more pay; in direct proportion.
This could be written: Earnings $\propto$ Hours worked
If you work 2 hours you get paid $40. If you work 3 hours you get paid $60, etc. . .

Constant of Proportionality

The "constant of proportionality" is the value that relates the two amounts
Example: You are paid $20 an hour, what is the constant of proportionality?
The constant of proportionality is 20 because: Earnings = 20 × Hours worked
This can be written: $y = kx$
Where k is the constant of proportionality

Example: y is directly proportional to x, and when x=3 then y=15. What is the constant of proportionality?
Solution: They are directly proportional, so: $y = kx$
Put in what we know (y=15 and x=3): $15 = k \times 3$ $k = 5$
The constant of proportionality is 5: $y = 5x$

When we know the constant of proportionality we can then answer other questions.
Example: What is the value of y when x = 9?
Solution: $y = 5 \times 9 = 45$

Example: What is the value of x when y = 2?
Solution: $2 = 5x$ $x = 2/5 = 0.4$

Inversely Proportional

Inversely Proportional: when one value decreases at the same rate that the other increases. Speed and travel time are Inversely Proportional because the faster we go the shorter the time. As speed goes up, travel time goes down. And as speed goes down, travel time goes up.

This is the same thing as: y is inversely proportional to x and y is directly proportional to 1/x. Which can be written $y = 1/x$

Example: 4 people can paint a fence in 3 hours. How long will it take 6 people to paint it? (Assume everyone works at the same rate.) It is an Inverse Proportion: As the number of people goes up, the painting time goes down. As the number of people goes down, the painting time goes up.
We can use: $t = k/n$
Where: t = number of hours, k = constant of proportionality, n = number of people
"4 people can paint a fence in 3 hours" means that t = 3 when n = 4
$3 = k/4$ so $k = 12$
So now we know: $t = 12/n$
And when n = 6: $t = 12/6 = 2$ hours So 6 people will take 2 hours to paint the fence.

Example: How many people are needed to complete the job in half an hour?
½ = 12/n n = 12 / ½ = 24 So it needs 24 people to complete the job in half an hour. (Assuming they don't all get in each other's way!)

Proportional to ...
It is also possible to be proportional to a square, a cube, an exponential, or other function!
Example: A stone is dropped from the top of a high tower. The distance it falls is proportional to the square of the time of fall. The stone falls 19.6 m after 2 seconds, how far does it fall after 3 seconds?

We can use: $d = kt^2$ Where: d is the distance fallen and t is the time of fall

When d = 19.6 then t = 2 $19.6 = k \times 2^2$ 19.6 = 4k k = 4.9 So now we know: d = 4.9t2
And when t = 3: $d = 4.9 \times 32$ d = 44.1 So it has fallen 44.1 m after 3 seconds.

Sample Questions:
11. If c is directly proportional to s^2 and $c = \frac{7}{16}$ when $s = \frac{1}{4}$, what is the value of s when c = 175?

12. If a is inversely proportional to b and a = 36 when b = 12, what is the value of a when b = 48?

13. If y is directly proportional to x and given y = 9 when x = 5, find:
a) the value of y when x = 15
b) the value of x when y = 6

14. Jane ran 100 meters in 15 seconds. How long did she take to run 2 meter?

15. A car travels 125 miles in 3 hours. How far would it travel in 5 hours?

RATE PROBLEM USING VARIABLE MANIPULATION

Example: A passenger train leaves the train depot 2 hours after a freight train left the same depot. The freight train is traveling 20 mph slower than the passenger train. Find the rate of each train, if the passenger train overtakes the freight train in three hours.

Solution: First, interpret the word problem and make a table:

The passenger train drove for three hours to catch up to the freight train; "3". But note that the freight train had a two-hour head start. That means that the freight train was going for five hours.

	d	r	t
passenger train	d = 3r	r	3
freight train	d = 5(r – 20)	r – 20	3 + 2 = 5
total	---	---	---

Now that I have this information, I can try to find my equation. Using the fact that d = rt, the first row gives: d = 3r. The second row gives: d = 5(r – 20)

Since the distances are equal, set the equations equal: 3r = 5(r – 20)

Solving gives r = 50—passenger train is going 50 mph

That means the freight train is going 50 – 20 = 30 mph

Sample Problems:

16. A car and a bus set out at 2 p.m. from the same point, headed in the same direction. The average speed of the car is 30 mph slower than twice the speed of the bus. In two hours, the car is 20 miles ahead of the bus. Find the rate of the car.

17. A dog eats 7 cans of food in 3 days. At this rate, how many cans of food does the dog eat in 3 + *d* days?

18. Harry is a piano student who can learn 2 new pieces of music in a week. If his piano teacher gives him 3 new pieces every week for 4 weeks, how many weeks will it take Harry to learn all these pieces?

19. Vehicle A averages 14 miles per gallon of gasoline, and Vehicle B averages 36 miles per gallon of gasoline. At these rates, how many more gallons of gasoline does Vehicle A need than Vehicle B to make a 1,008-mile trip?

20. A hot-air balloon 70 meters above the ground is falling at a constant rate of 60 meters per second while another hot-air balloon 10 meters above the ground is rising at a constant rate of 15 meters per second. To the nearest tenth of a second, after how many seconds will the 2 balloons be the same height above the ground?

RATE PROBLEMS WITH DATA

The shipping rate for customers of Ship Quick consists of a fee per box and a price per pound for each box. The table below gives the fee and the price per pound for customers shipping boxes of various weights.

Weight of box (pounds)	Fee	Price per pound
Less than 10	$ 5.00	$1.00
10–25	$ 10.00	$0.65
More than 25	$ 20.00	$0.30

21. Gregg wants Ship Quick to ship 1 box that weighs 15 pounds. What is the shipping rate for this box?

22. Holly wants to ship 2 boxes. The first box weighs 12.5 lbs and the second box weights 28 lbs. What is the shipping rate for both boxes?

23. April wants to know if it would be less expensive to ship her books in one box or two separate boxes. Her total weight is 50 lbs.

24. Kaylee is planning to purchase a car. She will need to borrow some of the money and has a chart, shown below, to use to approximate her monthly payment. The chart gives the approximate monthly payment per $1,000 borrowed.

Monthly payment per $1,000 borrowed for various annual rates and various numbers of payments			
	Number of monthly payments		
Annual interest rate	36	48	60
5%	$29.97	$23.03	$18.87
8%	$31.34	$24.41	$20.28
10%	$32.27	$25.36	$21.24
12%	$33.22	$26.34	$22.24

A local dealership is having an end-of-the-model-year clearance sale and is offering 5% annual interest on new-car loans for 36, 48, or 60 months. The maximum amount Kaylee can budget for her monthly car payment is $300. Of the following loan amounts, which one is the maximum Kaylee can borrow at 5% annual interest and stay within her budget?

a. $10,000
b. $13,000
c. $14,000
d. $15,000
e. $20,000

AVERAGE RATE

Average rate is *not* simply the average of the rates. It's the average of the total amounts. The most common rate is speed-distance over time-and the most common question about average rates is average speed-total distance over total time.

Example: A 555-mile, 5-hour plane trip was flown at two speeds. For the first part of the trip, the average speed was 105 mph. Then the tailwind picked up, and the remainder of the trip was flown at an average speed of 115 mph. For how long did the plane fly at each speed?

Solution: First, set up a grid:

	d	r	t
first part	d	105	t
second part	555 – d	115	5 – t
total	555	---	5

Using "d = rt", the first row gives me d = 105t and the second row gives: 555 – d = 115(5 – t)

Since the two distances add up to 555, add the two distance expressions, and set their sum equal to the given total: 555 = 105t + 115(5 – t) 2 = t

According to the grid, "t" stands for the time spent on the first part of the trip, so the answer is "The plane flew for two hours at 105 mph and three hours at115 mph."

Remember: You can add distances and you can add times, but you cannot add rates. Think about it: If you drive20 mph on one street, and 40 mph on another street, does that mean you averaged 60 mph?

Sample Questions:

25. If the first 120 miles of a journey is at 40 mph and the next 120 miles is at 60 mph, what is the average speed?

26. Marion spent all day on a sightseeing trip in Tuscany. First she boarded the bus which went 15mph through a 30 mile section of the countryside. The bus then stopped for lunch in Florence before continuing on a 3 hour tour of the city's sights at speed of 10mph. Finally, the bus left the city and drove 40 miles straight back to the hotel. Marion arrived back at her hotel exactly 2 hours after leaving Florence. What was the bus's average rate for the entire journey?

27. Tracey ran to the top of a steep hill at an average pace of 6 miles per hour. She took the exact same trail back down. To her relief, the descent was much faster; her average speed rose to 14 miles per hour. If the entire run took Tracey exactly one hour to complete and she did not make any stops, what is the length of trail in miles one way?

THE WEIGHTED AVERAGE

ACT writers have a particular fondness for "weighted average" problems. First, let's look at a regular unweighted average question.

Regular Average Example: If Sally received a grade of 90 on a test last week and a grade of 100 on a test this week, what is her average for the two tests? Piece of cake, right? The answer is 95. You added the scores and divided by 2.

Now let's turn the same question into a weighted average question.

Weighted Average Example: If Sally's average for the entire year last year was 90, and her average for the entire year this year was 100, is her average for the two years combined equal to 95? The answer is "not necessarily." If Sally took the same number of courses in both years, then yes, her average is 95. But what if last year she took 6 courses while this year she took only 2 courses? Can you compare the two years equally? ACT likes to test your answer to this question.

To solve weighted average problems, break it down by using the pie method or use a table to help organize the information and write an equation for the solution as follows.

Pie Method Example: The starting team of a baseball club has 9 members who have an average of 12 home runs apiece for the season. The second-string team for the baseball club has 7 members who have an average of 8 home runs apiece for the season. What is the average number of home runs for the starting team and the second-string team combined?

Pie Method Solution: The ACT test writers want to see whether you spot this as a weighted average problem. If you thought the first-string team was exactly equivalent to the second-string team, then you merely had to take the average of the two averages, 12 and 8, to get 10. In weighted average problems, the ACT test writers always include the average of the two averages among the answer choices, and it is always wrong so cross it off at the beginning.

The two teams are not equivalent because there are different numbers of players on each team. To get the true average, we'll have to find the total number of home runs and divide by the total number of players. How do we do this? By going to the trusty average formula as usual. The first line of the problem says that the 9 members on the first team have an average of 12 runs apiece. So the sum of everything is 9 x 12, or 108.

The second sentence says that the 7 members of the second team have an average of 8 runs each.
So the total is 7 x 8, or 56.

Now we can find the true average. Add all the runs scored by the first team to all the runs scored by the second team: 108 + 56 = 164. This is the true total. We divide it by the total number of players (9 + 7 = 16). The answer is 10.25.

Table/Equation Example: An online seed supplier packages a seed mix that costs the company $30.30 per pound. The mix includes poppy seeds costing $35.65 per pound and clover seeds costing $8.90 per pound. If a worker is going to prepare some of this mix and has already measured out 15 pounds of poppy seeds, what quantity of clover seeds should he add? *Write your answer as a whole number or as a decimal rounded to the nearest tenth.*

Table/Equation Solution: Step 1: Define a variable and make a table. Let x represent the quantity of poppy seeds. Now make a table. Start by filling in the first two columns with what you know.

	Price per pound	Number of pounds	Total price
Poppy seeds	35.65	15	
Clover seeds	8.9	x	
Seed mix	30.3		

Next, finish filling in the second column. Since there are 15 pounds of poppy seeds and x pounds of clover seeds, there are a total of $15 + x$ pounds of seed mix. Fill in $15 + x$.

	Price per pound	Number of pounds	Total price
Poppy seeds	35.65	15	
Clover seeds	8.9	x	
Seed mix	30.3	$15 + x$	

Finally, multiply across to fill in the third column.

	Price per pound	Number of pounds	Total price
Poppy seeds	35.65	15	35.65(15)
Clover seeds	8.9	x	$8.9x$
Seed mix	30.3	$15 + x$	$30.3(15 + x)$

Step 2: Write an equation and solve. Using the information in the table, write an equation.
Total price of poppy seeds plus total price of clover seeds equals total price of seed mix.
$35.65\ (15) + 8.9x = 30.3(15 + x)$
Solve the equation
$35.65(15) + 8.9x = 30.3(15 + x)$
$534.75 + 8.9x = 454.5 + 30.3x$
$534.75 – 21.4x = 454.5$
$-21.4 = -80.25$
$X \approx 3.8$
The worker should add about 3.8 pounds of clover seeds.

Example: In a certain club, the average age of the male members is 35, and the average of the female members is 25. If 20% of the members are male, what is the average age of all the club members?

a. 26
b. 27
c. 28
d. 29
e. 30

Solution: The overall average is not simply the average of the two average ages. Because there is a lot more women than men, women carry more weight, and the overall average will be closer to 25 than 35. Pick particular numbers for the females and males, say 4 females and 1male. The ages of the 4 females total 4 times 25, or 100, and the age of the 1 male totals 35. The average then, is (100 + 35) divided by 5, or 27 or Answer B.

Sample Questions:

28. Pamela makes sachets using dried flowers and herbs. Her special blend costs $10.70 per kilogram, combining dried lavender at $14.15 per kilogram and dried chamomile at $7.25 per kilogram. Pamela is preparing a new batch of the blend and has measured out 2 kilograms of lavender. How much chamomile should she add to complete her sachets?

29. Duncan works for a company that manufactures liquid dyes for clothing. He currently wants to mix up some 50%-concentrated white dye. He has 24 gallons of 56%-concentrated white dye, as well as plenty of 36%-concentrated white dye. How many gallons of the 36%-concentrated white dye will Duncan need to add to the 56%-concentrated white dye to make a batch with a concentration of 50%?

30. For a college chemistry experiment, students need to prepare a solution containing ethylene glycol. They are each given 16 fluid ounces of a solution containing 13% ethylene glycol, to which they will add another solution that contains 19% ethylene glycol. How much of the 19% solution should they add to obtain a 16% ethylene glycol solution?

31. A factory is using eugenol, a compound extracted from cinnamon and cloves, in some of its products. The factory has just 15 gallons of oil that contains 84% eugenol, in addition to plenty of oil containing 59% eugenol. How many gallons of the oil with 59% eugenol should be added to the oil with 84% to obtain a product with 74% eugenol%?

32. A candy maker is mixing up some sugar syrup for use in making sweets. How much of 49% sugar syrup should she add to 4 liters of 9% sugar syrup to obtain a mixture that contains 45% sugar?

33. A landscaper wants to use a mixture of soil and sand to ensure proper water drainage. He has just 47 cubic meters of soil that is 32% sand and an unlimited quantity of soil that is 38% sand. How many cubic meters of the 38% soil should he add to the 32% to create a mixture that is 35% sand?

FINDING THE MISSING NUMBER

To find a missing number when you're given the average, **use the sum.** If the average of four numbers is 7, then the sum of those four numbers is 4 x 7, or 28. Suppose that three of the numbers are 3, 5, and 8. These numbers add up to 16 of that 28, which leaves 12 for the fourth number.

Example: Martin's average score after four tests is 89. What score on the fifth test would bring Martin's average up to exactly 90?

Solution: The best way to deal with changing averages is to go by the way of the sums. Use the old average to figure out the total of the first 4 scores:

Sum of first 4 scores = 4 x 89 = 356 And use the new average to figure out the total he needs after the fifth score.

Sum of 5 scores = 4 x 90 = 450 To get his sum up from 456 to 450, Martin needs to score 450 – 356 = 94

Sample Questions:

34. Tom has taken 5 of the 8 equally weighted tests in his U.S. History class this semester, and he has an average score of exactly 78.0 points. How many points does he need to earn on the 6th test to bring his average score up to exactly 80.0 points?

35. Marlon is bowling in a tournament and has the highest average after 5 games, with scores of 210, 225, 254, 231, and 280. In order to maintain this exact average, what *must* be Marlon's score for his 6th game?

36. Chris is on a mountain bike team that is earning money to attend the state competition. There are 15 members on the team and they have earned $525 so far. If the original cost per team member for the trip is $75, how much more does each team member need to earn in order to pay for the trip?

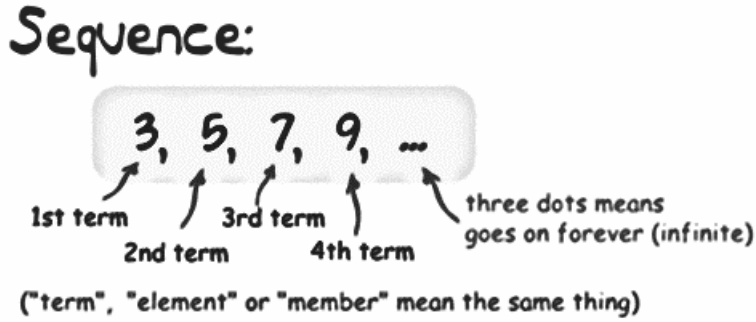
37. In Kara's math class, all tests count equally. So far, Kara has taken 4 of the 5 tests in math class this marking period and earned scores of 88%, 95%, 86% and 79% respectively. What is the minimum score Kara needs on the third test to have a test average of 85%?

38. Andy has grades of 84, 65, and 76 on three math tests. What grade must he obtain on the next test to have an average of exactly 80 for the four tests?

39. Linda computed the average of her six biology test scores by mistakenly adding the totals of five scores and dividing by five, giving her an average score of 88. When Linda realized her error, she recalculated and included the sixth test score of 82. What is the average of Linda's six biology tests?

SEQUENCES

A sequence is a set of things (usually numbers) that are in order.



ARITHMETIC SEQUENCES

In an Arithmetic Sequence the difference between one term and the next is a constant. In other words, we just **add** the same value each time ... infinitely.

For illustration purposes, look at the sequence: 1, 4, 7, 10, 13, 16, 19, 22, 25, ...
This sequence has a difference of 3 between each number. In general, we could write an arithmetic sequence like this:
{a, a+d, a+2d, a+3d, ... }
where: a is the first term, and d is the difference between the terms (called the "common difference")
In the example: a = 1 (the first term) and d = 3 (the "common difference" between terms)
And we get:
{a, a+d, a+2d, a+3d, ... }
{1, 1+3, 1+2×3, 1+3×3, ... }
{1, 4, 7, 10, ... }

EXPLICIT FORMULA FOR ARITHMETIC SEQUENCES

We can write an Arithmetic Sequence as an explicit formula:
$a_n = a_1 + d(n-1)$
(We use "n-1" because d is not used in the 1st term).

Example: Write the explicit formula for arithmetic sequence, and calculate the 4th term for
3, 8, 13, 18, 23, 28, 33, 38, ...
Solution: This sequence has a difference of 5 between each number.
The values of a and d are: a = 3 (the first term) and d = 5 (the "common difference")
The explicit formula can be calculated:
$a_n = a_1 + d(n-1)$
$= 3 + 5(n-1)$
$= 3 + 5n – 5$
$= 5n - 2$
So, the 4th term is: $a_4 = 5(4) - 2 = 18$

RECURSIVE FORMULA FOR ARITHMETIC SEQUENCES

We can also write an Arithmetic Sequence as a recursive formula:

$a_n = a_{n-1} + d$

We use "n-1" because d is not used in the 1st term).

Example: Write the recursive formula for arithmetic sequence, and calculate the 4th term for

3, 8, 13, 18, 23, 28, 33, 38, ...

Solution: As from above , this sequence has a difference of 5 between each number.

The values of a and d are: a = 3 (the first term) and d = 5 (the "common difference")

The recursive formula can be calculated:

$a_n = a_{n-1} + d$

$a_2 = a_1 + 5 = 3 + 5 = 8$

$a_3 = a_2 + 5 = 8 + 5 = 13$

So, the 4th term is: $a_4 = a_3 + 5 = 13 + 5 = 18$

Arithmetic Sequences are sometimes called Arithmetic Progressions (A.P.'s)

Sample Question:

40. What is the fourth term in the arithmetic sequence 13, 10, 7, ...?

41. Which of the following statements is NOT true about the arithmetic sequence 16, 11, 6, 1. . .?
    a. The fifth term is -4.
    b. The sum of the first 5 terms is 30.
    c. The seventh term is -12
    d. The common difference of consecutive integers is -5.
    e. The sum of the first 7 terms is 7.

42. The greatest integer of a set of consecutive even integers is 12. If the sum of these integers is 40, how many integers are in this set?

43. What is the sum of the first 4 terms of the arithmetic sequence in which the 6th term is 8 and the 10th term is 13?

GEOMETRIC SEQUENCE

In a Geometric Sequence each term is found by **multiplying** the previous term by a constant.
For illustration, look at the sequence: 2, 4, 8, 16, 32, 64, 128, 256, . . .
This sequence has a factor of 2 between each number.
Each term (except the first term) is found by multiplying the previous term by 2.

In general, we write a Geometric Sequence like this: $\{a, ar, ar^2, ar^3, ... \}$
where: a is the first term, and r is the factor between the terms (called the "common ratio")

Example: {1,2,4,8,...}
Solution: The sequence starts at 1 and doubles each time, so a=1 (the first term) and r=2 (the "common ratio" between terms is a doubling)
And we get:
$\{a, ar, ar^2, ar^3, ... \}$
$= \{1, 1\times2, 1\times22, 1\times23, ... \}$
$= \{1, 2, 4, 8, ... \}$
But be careful, r should not be 0:
When r = 0, we get the sequence {a, 0, 0,...} which is not geometric

EXPLICIT FORMULA FOR GEOMETRIC SEQUENCES

We can also calculate any term using the explicit formula:
$a_n = a_1r^{(n-1)}$
(We use "n-1" because ar^0 is for the 1st term)

Example: 10, 30, 90, 270, 810, 2430, ...
Solution: This sequence has a factor of 3 between each number.
The values of a and r are: a = 10 (the first term) and r = 3 (the "common ratio")
The Rule for any term is:
$a_n = 10 \times 3^{(n-1)}$
So, the 4th term is: $a_4 = 10 \times 3^{(4-1)}$
$= 10 \times 3^3$
$= 10 \times 27 = 270$

And the 10th term is: $a_{10} = 10 \times 3^{(10-1)}$
$= 10 \times 3^9$
$= 10 \times 19683 = 196830$

RECURSIVE FORMULA FOR GEOMETRIC SEQUENCES

We can also calculate a geometric sequence using the recursive formula:
$a_n = a_{n-1}(r)$

Example: 10, 30, 90, 270, 810, 2430, ...
Solution: As listed above, this sequence has a factor of 3 between each number.
The values of a and r are: a = 10 (the first term) and r = 3 (the "common ratio")
$a_2 = a_1(r) = 10(3) = 30$
$a_3 = a_2(r) = 30(3) = 90$
$a_4 = a_3(r) = 90(3) = 270$

A Geometric Sequence can also have smaller and smaller values:
Example: 4, 2, 1, 0.5, 0.25, ...
Solution: This sequence has a factor of 0.5 (a half) between each number.
Using the explicit formula: $a_n = 4 \times (0.5)^{n-1}$
Using the recursive formula: $a_n = a_{n-1}(0.5)$

Geometric Sequences are sometimes called Geometric Progressions (G.P.'s)

Example Questions:
44. What is the next term after -1/3 in the geometric sequence 9, -3, 1, -1/3, . . . ?

45. Which of the following statements is NOT true about the geometric sequence 26, 18, 9, . . . ?
    a. The fourth term is 4.5
    b. The sum of the first five terms is 59.75
    c. Each consecutive term is ½ of the previous term
    d. Each consecutive term is evenly divisible by 3
    e. The common ratio of consecutive terms is 2:1

46. The first term is 1 in the geometric sequence 1, -3, 9, -27, ... What is the SEVENTH term in the geometric sequence?

AVERAGE OF EVENLY SPACED NUMBERS

When you have a set of evenly spaced numbers, such as 10 consecutive numbers, or all the odd numbers between 5 and 35, you can use a shortcut rather than the averages formula. To find the average of evenly spaced numbers, just **average the smallest and the largest.** The average of all the integers from 13 through 77 is the same as the average of 13 and 77.

$$\frac{13+77}{2} = \frac{90}{2} = 45.$$

Sample Questions:

47. What is the average of all integers from 15 through 99?

48. What is the difference between the average of all consecutive numbers from 33 through 99 and the average of all even numbers from 2 through 24?

49. What is the average of the set {3, 5, 7, 9, 11, 13, 15, 17, 19, 21}?

REPEATING DECIMAL

To find a particular digit in a repeating decimal, note the **number of digits in the cluster that repeats.** If there are two digits in that cluster, then every second digit is the same. If there are three digits in that cluster, then every third digit is the same. And so on. For example, the decimal equivalent of 1/27 is .037037037...which is best written $\overline{0.037}$.

There are three digits in the repeating cluster, so every third digit is the same: 7. To find the 50th digit, look for the multiple of 3 just less than 50-that's 48. The 48th digit is 7, and with the 49th digit the pattern repeats with 0. So, the 50th digit is 3.

Sample Questions:

50. In the number 0.1666, what digit is coming next?

51. In the number, 0.191919, what digit is coming next?

52. What is the 217th digit after the decimal point in the repeating decimal $0.\overline{3456}$?

53. When 4/11 is written as a decimal, what is the 100th digit after the decimal point?

OTHER REPEATING NUMBERS

This idea of repeating can be used in other number examples.

Example: If the first day of the year is a Monday, what is the 260th day?
Solution: Sketch out a little calendar until you see a pattern: Day 1 is Monday, 2 is Tuesday, 3 is Wednesday, 4, Thursday, 5, Friday, 6 Saturday, 7 Sunday, 8 Monday, and so on. Notice that Sundays are always multiples of 7. Pick a multiple of 7 close to 260, such as 259. That means Day 269 is a Sunday, so Day 260 is one more so a Monday.

SOLVING PROBLEMS USING THE SLOT METHOD

Combination problems ask you how many different ways a number of things could be chosen or combined. The rules for combination problems on the ACT are straightforward.

- Figure out the number of slots you need to fill.
- Fill in those slots.
- Find the product.

On a more difficult problem, you may run into a combination with more restricted elements. Just be sure to read the problem carefully before attempting it. If the question makes your head spin, leave it and return to it later, or pick your Letter of the Day and move on. For good measure, though, here's what one of those tougher ones might look like.

Example: At the school cafeteria, 2 boys and 4 girls are forming a lunch line. If the boys must stand in the first and last places in line, how many different lines can be formed?

Solution: These restrictions might make this problem seem daunting, but this is where the slot method is really helpful. We have six spots in line to fill, so draw six slots:

___ ___ ___ ___ ___ ___

Fill in the restricted spots first. The problem tells us that the two boys must stand in the first and last places in line. This means that either of the boys could stand in first place, and then the other boy will stand in last. This means that we have two options for the first place, but only one for the last.

_2_ ___ ___ ___ ___ _1_

Now do the same with the unrestricted parts. Any of the four girls could stand in the second spot.

_2_ _4_ ___ ___ ___ _1_

Now, since one of the girls is standing in the second spot, there are only three left to stand in the third spot, and so on, and so on.

_2_ _4_ _3_ _2_ _1_ _1_

Now, as ever, just take the product, co find that the correct answer is choice (C), 48.

_2_ x _4_ x _3_ x _2_ x _1_ x _1_ = 48

Sample Questions:

54. In the word HAWKS, how many ways is it possible to rearrange the letters if none repeat and the letter W must go last?

55. A classroom has 10 tables that will seat up to 4 students each. If 20 students are seated at tables, and NO tables are empty, what is the greatest possible number of tables that could be filled with students?

56. How many different positive three-digit integers can be formed if the three digits 3, 4, and 5 must be used in each of the integers?

COUNTING CONSECUTIVE INTEGERS

To find the number of consecutive integers between two values, **subtract the smallest from the largest and add 1.** So to find the number of consecutive integers from 13 through 31, subtract: 31 – 13 = 18. Then add 1: 18 + 1 = 19. There are 19 consecutive integers from 13 through 31.

USING THE SLOT METHOD TO FIND CONSECUTIVE INTEGERS

First write the sequence including the unknown numbers by leaving slots to fill in the numbers. Then find the "common difference" between the first and last terms in the sequence. Divide the difference by the number of "jumps" between the first and last integer.

Example: What two numbers should be placed in the blanks below so that the difference between successive entries is the same?

26, _____, _____, 53

a. 36, 43
b. 35, 44
c. 34, 45
d. 33, 46
e. 30, 49

Solution: Common Difference = 53-26 = 27. There are three jumps (26 to 2nd term, 2nd term to 3rd term, 3rd term to 53) so divide 27 by 3 = 9. The second term would be 26 + 9 = 35. The third term would be 35 + 9 = 44 (B).

USING ARITHMETIC SEQUENCE TO FIND CONSECUTIVE INTEGERS

Example: What two numbers should be placed in the blanks below so that the difference between successive entries is the same?

26, _____, _____, 53

Solution: In the arithmetic sequence, you can think of the terms as 26, 26 + s, 26 + s + s, and 26 + s + s + s. In the example, s represents the difference between successive terms. The final term is 53, so set up an algebraic equation: 26 + s + s + s = 53. Solve the equation for s, by first combining like terms: 26 + 3s = 53. Subtract 26 from both sides to get 3s = 27. Divide both sides by 3 to find that s, the difference between terms is 9. Therefore, the terms are 26, 26 + 9, 26 + 9 + 9, and 53 or 26, 35, 44, and 53 (B)

Sample Questions:

57. For the difference between consecutive numbers to be the same, which 3 numbers must be placed in the blanks below?

11, _____, _____, _____, 47

58. What two numbers should be placed in the blanks below so that the difference between the consecutive numbers is the same?

13, ___, ___, 34

Answer Key

1. 68%
2. 156%
3. $22.29
4. 2/9
5. 2/5
6. There are presently 1,620 men at the college.
7. 90 additional women
8. 3:1
9. 15:4
10. There are 52 more girls than boys in the school.
11. 5
12. 9
13. A) 27 and B)3.333
14. 0.3 seconds
15. 208.33 miles
16. 50 mph
17. $7 + \frac{7}{3}d$
18. 6 weeks
19. 44 gallons
20. 0.8 seconds
21. $19.75
22. $46.53
23. 1 box
24. $15,000
25. 48 mph
26. 11.43 mph
27. 4.2 miles
28. Pamela should add 2 kilograms of chamomile to complete the sachets.
29. Duncan should add about 10.3 gallons of the 36%-concentrated white dye.
30. To obtain a 16% ethylene glycol solution, the students should add 16 fluid ounces of the 19% solution.
31. To obtain the desired product, 10 gallons of the oil containing 59% eugenol should be added.
32. To obtain a mixture that contains 45% sugar, 36 liters of the 49% sugar syrup should be added.
33. The landscaper should add 47 cubic meters of the soil that is 38% sand.
34. 90
35. 240
36. $40.00
37. 77%
38. 95%
39. 87%
40. 4
41. C
42. 5 numbers
43. 14.5
44. 1/9
45. D
46. 729
47. 42
48. 22
49. 9
50. 6
51. 1
52. 3
53. 6
54. 24
55. 3
56. 6
57. 20, 29, 38
58. 20, 27