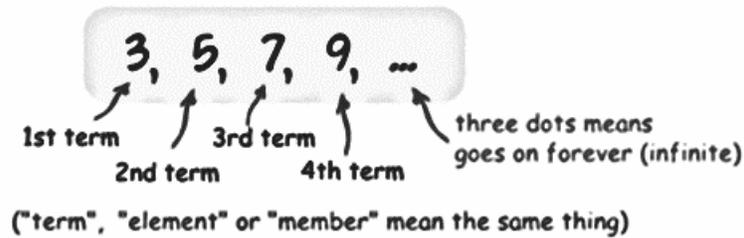


Math 1 Proportion & Probability Part 3 Sequences, Number Patterns & Probability

SEQUENCES

A sequence is a set of things (usually numbers) that are in order.

Sequence:



ARITHMETIC SEQUENCES

In an Arithmetic Sequence the difference between one term and the next is a constant. In other words, we just **add** the same value each time ... infinitely.

For illustration purposes, look at the sequence: 1, 4, 7, 10, 13, 16, 19, 22, 25,

This sequence has a difference of 3 between each number. In general, we could write an arithmetic sequence like this:

$$\{a, a+d, a+2d, a+3d, \dots\}$$

where: a is the first term, and d is the difference between the terms (called the "common difference")

In the example: $a = 1$ (the first term) and $d = 3$ (the "common difference" between terms)

And we get:

$$\{a, a+d, a+2d, a+3d, \dots\}$$

$$\{1, 1+3, 1+2 \times 3, 1+3 \times 3, \dots\}$$

$$\{1, 4, 7, 10, \dots\}$$

EXPLICIT FORMULA FOR ARITHMETIC SEQUENCES

We can write an Arithmetic Sequence as an explicit formula:

$$a_n = a_1 + d(n-1)$$

(We use "n-1" because d is not used in the 1st term).

Example: Write the explicit formula for arithmetic sequence, and calculate the 4th term for 3, 8, 13, 18, 23, 28, 33, 38,

Solution: This sequence has a difference of 5 between each number.

The values of a and d are: $a = 3$ (the first term) and $d = 5$ (the "common difference")

The explicit formula can be calculated:

$$a_n = a_1 + d(n-1)$$

$$= 3 + 5(n-1)$$

$$= 3 + 5n - 5$$

$$= 5n - 2$$

So, the 4th term is: $a_4 = 5(4) - 2 = 18$

GEOMETRIC SEQUENCE

In a Geometric Sequence each term is found by **multiplying** the previous term by a constant.

For illustration, look at the sequence: 2, 4, 8, 16, 32, 64, 128, 256, . . .

This sequence has a factor of 2 between each number.

Each term (except the first term) is found by multiplying the previous term by 2.

In general, we write a Geometric Sequence like this:

$$\{a, ar, ar^2, ar^3, \dots\}$$

where: a is the first term, and r is the factor between the terms (called the "common ratio")

Example: $\{1, 2, 4, 8, \dots\}$

Solution: The sequence starts at 1 and doubles each time, so $a=1$ (the first term) and $r=2$ (the "common ratio" between terms is a doubling)

And we get:

$$\{a, ar, ar^2, ar^3, \dots\}$$

$$= \{1, 1 \times 2, 1 \times 2^2, 1 \times 2^3, \dots\}$$

$$= \{1, 2, 4, 8, \dots\}$$

But be careful, r should not be 0:

When $r = 0$, we get the sequence $\{a, 0, 0, \dots\}$ which is not geometric

EXPLICIT FORMULA FOR GEOMETRIC SEQUENCES

We can also calculate any term using the explicit formula:

$$a_n = a_1 r^{(n-1)}$$

(We use " $n-1$ " because ar^0 is for the 1st term)

Example: 10, 30, 90, 270, 810, 2430,

Solution: This sequence has a factor of 3 between each number.

The values of a and r are: $a = 10$ (the first term) and $r = 3$ (the "common ratio")

The Rule for any term is:

$$a_n = 10 \times 3^{(n-1)}$$

$$\text{So, the 4th term is: } a_4 = 10 \times 3^{(4-1)}$$

$$= 10 \times 3^3$$

$$= 10 \times 27 = 270$$

$$\text{And the 10th term is: } a_{10} = 10 \times 3^{(10-1)}$$

$$= 10 \times 3^9$$

$$= 10 \times 19683 = 196830$$

RECURSIVE FORMULA FOR GEOMETRIC SEQUENCES

We can also calculate a geometric sequence using the recursive formula:

$$a_n = a_{n-1}(r)$$

Example: 10, 30, 90, 270, 810, 2430,

Solution: As listed above, this sequence has a factor of 3 between each number.

The values of a and r are: $a = 10$ (the first term) and $r = 3$ (the "common ratio")

$$a_2 = a_1(r) = 10(3) = 30$$

$$a_3 = a_2(r) = 30(3) = 90$$

$$a_4 = a_3(r) = 90(3) = 270$$

A Geometric Sequence can also have smaller and smaller values:

Example: 4, 2, 1, 0.5, 0.25,

Solution: This sequence has a factor of 0.5 (a half) between each number.

Using the explicit formula: $a_n = 4 \times (0.5)^{n-1}$

Using the recursive formula: $a_n = a_{n-1}(0.5)$

Geometric Sequences are sometimes called Geometric Progressions (G.P.'s)

Example Questions:

4. What is the next term after $-1/3$ in the geometric sequence 9, -3, 1, $-1/3$, . . . ?

5. Which of the following statements is NOT true about the geometric sequence 26, 18, 9, . . . ?

- a. The fourth term is 4.5
- b. The sum of the first five terms is 59.75
- c. Each consecutive term is $\frac{1}{2}$ of the previous term
- d. Each consecutive term is evenly divisible by 3
- e. The common ratio of consecutive terms is 2:1

6. The first term is 1 in the geometric sequence 1, -3, 9, -27, What is the SEVENTH term in the geometric sequence?

SOLVING PROBLEMS USING THE SLOT METHOD

Combination problems ask you how many different ways a number of things could be chosen or combined. The rules for combination problems on the ACT are straightforward.

- Figure out the number of slots you need to fill.
- Fill in those slots.
- Find the product.

On a more difficult problem, you may run into a combination with more restricted elements. Just be sure to read the problem carefully before attempting it. If the question makes your head spin, leave it and return to it later, or pick your Letter of the Day and move on. For good measure, though, here's what one of those tougher ones might look like.

Example: At the school cafeteria, 2 boys and 4 girls are forming a lunch line. If the boys must stand in the first and last places in line, how many different lines can be formed?

Solution: These restrictions might make this problem seem daunting, but this is where the slot method is really helpful. We have six spots in line to fill, so draw six slots:

___ ___ ___ ___ ___ ___

Fill in the restricted spots first. The problem tells us that the two boys must stand in the first and last places in line. This means that either of the boys could stand in first place, and then the other boy will stand in last. This means that we have two options for the first place, but only one for the last.

2 ___ ___ ___ ___ 1

Now do the same with the unrestricted parts. Any of the four girls could stand in the second spot.

2 4 ___ ___ ___ 1

Now, since one of the girls is standing in the second spot, there are only three left to stand in the third spot, and so on, and so on.

2 4 3 2 1 1

Now, as ever, just take the product, so find that the correct answer is choice (C), 48.

2 x 4 x 3 x 2 x 1 x 1 = 48

Sample Questions:

11. In the word HAWKS, how many ways is it possible to rearrange the letters if none repeat and the letter W must go last?
12. A classroom has 10 tables that will seat up to 4 students each. If 20 students are seated at tables, and NO tables are empty, what is the greatest possible number of tables that could be filled with students?
13. How many different positive three-digit integers can be formed if the three digits 3, 4, and 5 must be used in each of the integers?

COUNTING CONSECUTIVE INTEGERS

To find the number of consecutive integers between two values, **subtract the smallest from the largest and add 1**. So to find the number of consecutive integers from 13 through 31, subtract: $31 - 13 = 18$. Then add 1: $18 + 1 = 19$. There are 19 consecutive integers from 13 through 31.

USING THE SLOT METHOD TO FIND CONSECUTIVE INTEGERS

First write the sequence including the unknown numbers by leaving slots to fill in the numbers. Then find the “common difference” between the first and last terms in the sequence. Divide the difference by the number of “jumps” between the first and last integer.

Example: What two numbers should be placed in the blanks below so that the difference between successive entries is the same?

26, _____, _____, 53

- a. 36, 43
- b. 35, 44
- c. 34, 45
- d. 33, 46
- e. 30, 49

Solution: Common Difference = $53 - 26 = 27$. There are three jumps (26 to 2nd term, 2nd term to 3rd term, 3rd term to 53) so divide 27 by 3 = 9. The second term would be $26 + 9 = 35$. The third term would be $35 + 9 = 44$ (B).

USING ARITHMETIC SEQUENCE TO FIND CONSECUTIVE INTEGERS

Example: What two numbers should be placed in the blanks below so that the difference between successive entries is the same?

26, _____, _____, 53

Solution: In the arithmetic sequence, you can think of the terms as 26, $26 + s$, $26 + s + s$, and $26 + s + s + s$. In the example, s represents the difference between successive terms. The final term is 53, so set up an algebraic equation: $26 + s + s + s = 53$. Solve the equation for s , by first combining like terms: $26 + 3s = 53$. Subtract 26 from both sides to get $3s = 27$. Divide both sides by 3 to find that s , the difference between terms is 9. Therefore, the terms are 26, $26 + 9$, $26 + 9 + 9$, and 53 or 26, 35, 44, and 53 (B)

Sample Questions:

14. For the difference between consecutive numbers to be the same, which 3 numbers must be placed in the blanks below?

11, _____, _____, _____, 47

15. What two numbers should be placed in the blanks below so that the difference between the consecutive numbers is the same?

13, _____, _____, 34

PROBABILITY

Probability is always the number of desired outcomes divided by the number of possible outcomes. It can be expressed as a fraction, or sometimes, as a decimal.

$$\text{Probability} = \frac{\text{Favorable outcomes}}{\text{Total possible outcomes}}$$

If you have 12 shirts in a drawer and 9 of them are white, the probability of picking a white shirt at random is $\frac{9}{12} = \frac{3}{4}$. This probability can also be expressed as 0.75 or 75%.

Example: Herbie's practice bag contains 4 blue racquetballs, one red racquetball, and 6 green racquetballs. If he chooses a ball at random, which of the following is closest to the probability that the ball will NOT be green?

- 0.09
- 0.27
- 0.36
- 0.45
- 0.54

Solution:

Step 1: **Know the question.** Make sure you read carefully! We want the probability that the chosen ball will NOT be green.

Step 2: **Let the answers help.** Green balls account for slightly more than half the number of balls in the bag, so the likelihood that the ball will NOT be green could be slightly less than half. That eliminates choices (A), (B), and (E).

Step 3: **Break the problem into bite-sized pieces.** This is a pretty straightforward $\frac{\text{part}}{\text{whole}}$ problem. $\frac{\text{part}}{\text{whole}} = \frac{\text{NOT green}}{\text{all}} = \frac{5}{11}$.

The only slight difficulty is that the answers are not listed as fractions, but it's nothing a calculator can't help. Find $5 \div 11 = 0.45$, choice (D).

Sample Questions:

16. If a gumball is randomly chosen from a bag that contains exactly 7 yellow gumballs, 3 green gumballs, and 2 red gumballs, what is the probability that the gumball chosen is green?

17. There are 15 balls in a box: 7 balls are green, 3 are blue and 5 are white. Then 1 green and 1 blue balls are taken from the box and put away. What is the probability that a blue ball will NOT be selected at random from the box?

18. Seth has 4 plaid shirts and 5 solid-colored shirts hanging together in a closet. In his haste to get ready for work, he randomly grabs 1 of these 9 shirts. What is the probability that the shirt Seth grabs is plaid?

19. A partial deck of cards was found sitting out on a table. If the partial deck consists of 6 spades, 3 hearts, and 7 diamonds, what is the probability of randomly selecting a red card from this partial deck? (Note: diamonds and hearts are considered “red” while spades and clubs are considered “black”)?
20. A bag contains 6 red marbles, 5 yellow marbles, and 7 green marbles. How many additional red marbles must be added to the 18 marbles already in the bag so that the probability of randomly drawing a red marble is $\frac{3}{5}$?
21. A bag contains 12 red marbles, 5 yellow marbles, and 15 green marbles. How many additional red marbles must be added to the 32 marbles already in the bag so that the probability of randomly drawing a red marble is $\frac{3}{5}$?
22. To make a 750-piece jigsaw puzzle more challenging, a puzzle company includes 5 extra pieces in the box along with the 750 pieces, and those 5 extra pieces do not fit anywhere in the puzzle. If you buy such a puzzle box, break the seal on the box, and immediately select 1 piece at random, what is the probability that it will be 1 of the extra pieces?
23. An integer from 10 through 99, inclusive, is to be chosen at random. What is the probability that the number chosen will have 9 as at least 1 digit?

Answer Key

1. 4
2. C
3. 5 numbers
4. $\frac{1}{9}$
5. D
6. 729
7. 6
8. 1
9. 3
10. 6
11. 24
12. 3
13. 6
14. 20, 29, 38
15. 20, 27
16. $\frac{1}{4}$
17. $\frac{11}{13}$
18. $\frac{4}{9}$
19. $\frac{5}{8}$
20. 12
21. 18
22. $\frac{5}{755}$
23. $\frac{18}{90} = \frac{1}{5}$