

Math 1 Proportion & Probability Part 2

Average, Mean/Median/Mode, & Combinations

AVERAGE FORMULA

To find the average of a set of numbers, **add them up and divide by the number of numbers.**

$$\text{Average} = \frac{\text{Sum of the terms}}{\text{Number of the terms}}$$

Example: Find the average of the five numbers 12, 15, 23, 40, and 40.

Solution: First add them: $12 + 15 + 23 + 40 + 40 = 130$. Then divide the sum by 5: $130 \div 5 = 26$.

The average formula works for more complicated numbers: fractions, percentages, complicated numbers with variables, etc. To solve the more complicated problems, follow sound mathematical practices to get the correct answer.

Example: What is the average of $1/20$ and $1/30$?

Solution: To get the average, you must add the fractions and divide by two. Don't just average the denominators!!!

$$\text{Average} = \frac{\text{sum}}{\text{number of terms}} = \frac{\frac{1}{20} + \frac{1}{30}}{2} = \frac{\frac{3}{60} + \frac{2}{60}}{2} = \frac{\frac{5}{60}}{2} = \frac{\frac{1}{12}}{2} = \frac{1}{12} \times \frac{1}{2} = \frac{1}{24}$$

If you averaged the denominators you would have gotten $1/25$, but the answer is $1/24$. Don't be fooled!

Example: What is the average of the expressions $2x + 5$, $5x - 6$, $-4x + 2$?

- a. $x + 1/3$
- b. $x + 1$
- c. $3x + 1/3$
- d. $3x + 3$
- e. $3x + 3 \frac{1}{3}$

Solution: Don't let the algebraic terms confuse you, simply add them and divide by 3.

$$\text{Average} = \frac{\text{sum}}{\text{number of terms}} = \frac{(2x+5)+(5x-6)+(-4x+2)}{3} = \frac{(3x+1)}{3} = x + \frac{1}{3} \text{ or answer A}$$

Sample Questions:

1. What is the average of $2/3$ and $4/5$?
2. What is the average of $1/2$, $1/3$, and $1/4$?
3. What is the average of the expressions $x + 4$, $2x - 5$, $2x + 1$?
4. What is the average of the expressions $3x - 4$, $2x + 3$, $-2x - 1$?

USING THE AVERAGE TO FIND THE SUM

Since the average is just the sum of the numbers divided by the number of numbers, it follows that:

$$\text{Sum} = (\text{Average}) \times (\text{Number of terms})$$

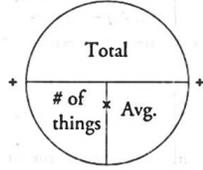
If the average of 10 numbers is 50, then they add up to 10×50 , or 500

Sample Questions:

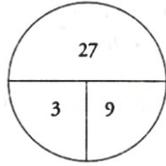
5. The average of 5 numbers is 77, what is the total of all five numbers?
6. A group of 25 numbers add up to 500. What is the average of these numbers?

AVERAGE: THE PIE METHOD

There are only three parts to any average question. The question must give you two of these parts, which are all you need to find the third. The average pie is an easy way to keep track of the information you get from questions dealing with averages. If you have the total, you can always divide by either the average or the number of things in the set (whichever you are given) to find the missing piece of the pie. Similarly, if you have the number of things and the average, you can multiply the two together to arrive at the total (the sum of all the items in the set).



For example, if you want to find the average of 9, 12, and 6 using the average pie, you know you have 3 items with a total of 27. Dividing the total, 27, by the number of things, 3, will yield the average, 9. Your pie looks like this



Although you probably could have done that without the average pie, more difficult average questions involve multiple calculations and lend themselves particularly well to using the pie.

Example: Over 9 games, a baseball team had an average of 8 runs per game. If the average number of runs for the first 7 games was 6 runs per game, and the same number of runs was scored in each of the last 2 games, how many runs did the team score during the last game?

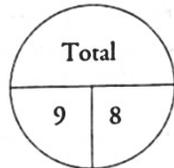
- 5
- 15
- 26
- 30
- 46

Solution:

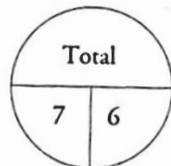
Step 1: Know the question. How many runs did the team score during the last game?

Step 2: Let the answers help. Eliminate choice (A). Even though the average for the first 7 is higher than all 9, the runs scored in the last two games can't be that few. Similarly, choice (E) is probably too big. If you don't trust your sense of numbers and you're not comfortable Ballparking here, however, leave both. It's a complicated question on a more advanced topic.

Step 3: Break the problem into bite-sized pieces. Use bite-size pieces to put the information from the first line of this problem into our trusty average pie.



What is the sum of everything for these 9 games? 9×8 , or 72. Now put the information from the second line into the average equation.



What is the sum of everything for these 7 games? 7×6 , or 42. If all 9 games added up to 72, and 7 of these games added up to 42, then the remaining 2 games added up to $72 - 42$, or 30. In case you are feeling smug about getting this far, the ACT writers made 30 the answer for choice (D). But of course you know that they only want the runs they scored in the last game. Because the same number of runs was scored in each of the last two games, the answer is $30/2$ or 15, choice (B).

Sample Questions:

7. The daily totals of enrollments at Jerry's cooking class last Monday through Friday were 24, 27, 23, 24, and 27. What was the average number of enrollments per day?

8. What is the average of $3/4$ and $4/5$?

9. As part of a school report on the cost of milk, Sammy wants to find the average cost of milk from local stores. She surveys 4 stores and finds the cost per gallon of whole milk from the 4 stores to be \$2.38, \$2.50, \$2.79, \$2.46, respectively. Using this data, what is the average cost of purchasing one gallon of whole milk from these 4 gas stores?

10. Timothy's average score on the first 4 tests was 76. On the next 5 tests his average score was 85. What was his average score on all 9 tests?

11. Tracy mowed lawns for 2 hours and earned \$7.40 per hour. Then she washed windows for 3 hours and earned \$6.50 per hour. What were Tracy's average earnings per hour for all 5 hours?

12. After taking 3 quizzes, your average is 72 out of 100. What must your average be on the next two quizzes so that on 5 quizzes you increase your average to 77?

13. If the average of 8, 11, 25, and p is 15, find p .

14. If $a = 3b = 6c$, what is the average of a , b and c in terms of a ?

WEIGHTED AVERAGE

Another type of average problem involves the weighted average - which is the average of two or more terms that do not all have the same number of members. To find the weighted term, multiply each term by its weighting factor, which is the number of times each term occurs.

The formula for weighted average is:

$$\text{Weighted Average} = \frac{\text{Sum of Weighted Terms}}{\text{Total Number of Terms}}$$

Example:

A class of 25 students took a science test. 10 students had an average score of 80. The other students had an average score of 60. What is the average score of the whole class?

Solution:

Step 1: To get the sum of weighted terms, multiply each average by the number of students that had that average and then sum them up.

$$80 \times 10 + 60 \times 15 = 800 + 900 = 1700$$

Step 2: Total number of terms = Total number of students = 25

Step 3: Using the formula

$$\begin{aligned} \text{Weighted Average} &= \frac{\text{Sum of Weighted Terms}}{\text{Total Number of Terms}} \\ &= \frac{1700}{25} \\ &= 68 \end{aligned}$$

Answer: The average score of the whole class is 68.

Be careful! You will get the wrong answer if you add the two average scores and divide the answer by two.

Weighted average is the way to calculate GPA (Grade Point Average).

Example:

First semester, Joanie got a 4 in Seminar (which counts for two class periods), 3 in chemistry, 3 in math, 4 in art, 4 in choir, 2 in novel writing and 2 in advanced fitness. What is Joanie's GPA?

Solution: Count the number of classes that Joanie received a 4 and multiply that together. Add that to the number of classes that Joanie received a 3 multiplied by 3. Add that by the number of classes that Joanie received a 2 and multiply that by 2. Then divide the total by 8 (the total number of classes Joanie took in first semester).

$$\frac{4(4) + 3(2) + 2(2)}{8} = 3.25$$

Weighted average is also used when calculating amounts in mixtures.

Example: Emily mixed together 9 gal. of Brand A fruit drink and 8 gal. of Brand B fruit drink which contains 48% fruit juice. Find the percent of fruit juice in Brand A if the mixture contained 30% fruit juice.

Solution: Emily mixed 9 gal of A with 8 gal of B which makes 17 gal of total juice. We are trying to calculate the % fruit juice in A, so that is labeled X. Add the amount of fruit juice in A and B and set that equal to the amount of juice in the final solution. Amount fruit juice is calculated as per weighted average: number of gallons times the percent fruit juice.

$$9x + 8(0.48) = 17(0.30)$$

$$9x + 3.84 = 5.1$$

$$9x = 5.1 - 3.84 = 1.26$$

$$x = 1.26/9 = 0.14 = 14\% \quad \text{There is 14\% fruit juice in Brand A}$$

Sample Questions:

15. Fifteen accounting majors have an average grade of 90. Seven marketing majors averaged 85, and ten finance majors averaged 93. What is the weighted mean for the 32 students?

16. 5 fl. oz. of a 2% alcohol solution was mixed with 11 fl. oz. of a 66% alcohol solution. Find the concentration of the new mixture.

17. 9 gal. of a sugar solution was mixed with 6 gal. of a 90% sugar solution to make a 84% sugar solution. Find the percent concentration of the first solution.

18. 7 L of an acid solution was mixed with 3 L of a 15% acid solution to make a 29% acid solution. Find the percent concentration of the first solution.

19. How many mg of a metal containing 45% nickel must be combined with 6 mg of pure nickel to form an alloy containing 78% nickel?

20. How many gal. of a 65% saline solution must be mixed with 8 gal. of pure water to make a 25% solution?

21. A feedlot in Carroll County has developed its own blend of grains for horses. The blend includes oats, which cost \$9.90 per kilogram, and feed corn, which costs \$4.40 per kilogram. Overall, the mixture costs \$8.25 per kilogram. If a worker at the feedlot is mixing up a batch and starts with 220 kilograms of oats, how many kilograms of corn should he add to make the blend?

Use the following table for the next 2 problems

| | | | |
|----|-----|----|-----|
| A | 4.0 | C | 2.0 |
| A- | 3.7 | C- | 1.7 |
| B+ | 3.3 | D+ | 1.3 |
| B | 3.0 | D | 1.0 |
| B- | 2.7 | F | 0.0 |
| C+ | 2.3 | | |

22. Calculate Hannah's GPA if she received the following grades: A, B-, A-, C, A-, B-, A, A.

23. Calculate Joey's GPA if he received the following grades: B+, B-, A-, B+, B, B-, A-, B.

24. James must have a 3.0 GPA to be able to play on the team. He already has a B-, B-, C, B-, A, A, B-. What is the minimum grade he needs to get in his last class to be able to play on the team?

AVERAGE RATE: DISTANCE RATE PROBLEMS

Distance = Rate x Time

The first important formula to memorize is: $D = R \times T$. This stands for Distance = Rate x Time. Think of it as the “DIRT” formula and writing it this way is an easy way to remember. It is perfectly acceptable to also think of it as $\text{Time} = \text{Distance} / \text{Rate}$ or as $\text{Rate} = \text{Distance} / \text{Time}$. Usually the “Rate” is speed but it could be anything “per” anything. In a word problem, if you see the word “per” you know this is a question involving rates.

Average rate is its own special concept and you will notice that it is NOT an Average of the Speeds (which would be something like the Sum of the Speeds / the Number of Different Speeds or what we know as the Arithmetic Mean). Average Rate is completely different. The most common rate is speed-distance over time-and the most common question about average rates is average speed-total distance over total time.

Average Rate = Total Distance / Total Time

Example: Ariella drove 40 miles to see her cousin and at 20 mph. The trip took Ariella 2 hours. Then, Ariella drove from her cousin’s house and drove another 30 miles to the store at a speed of 10mph. It took Ariella 3 hours to arrive at the store. What was Ariella’s average speed for the trip?

Solution: Average Speed = Total Distance / Total Time.

Ariella traveled 40 miles + 30 miles, so her Total Distance was 70 miles. She drove for 2 hours + 3 hours, so her Total Time was 5 hours. $70/5 = 14$.

Her Average Speed for the whole trip was 14 mph.

The Average Speed in this problem is 14 mph, which is different from the “Average of the Speeds.” If we had just averaged the two speeds (10mph and 20mph), we would have gotten 15mph. Think of Average Speed as a weighted average. Because Ariella spent more time in the problem going 10mph than 20mph, so it makes sense that the Average Speed would be closer to 10mph. Be careful! The “Average of the Speeds” will often be a tempting wrong answer choice on the ACT!

Example: Tracey ran to the top of a steep hill at an average pace of 6 miles per hour. She took the exact same trail back down. To her relief, the descent was much faster; her average speed rose to 14 miles per hour. If the entire run took Tracey exactly one hour to complete and she did not make any stops, what is the length of trail in miles one way?

Solution: For the way up the hill, we know that $D = 6\text{mph} \times T$.

For the way down the hill, we know that $D = 14\text{mph} \times T$. Since we went know that the distance up the hill was the same as the distance down the hill, we can pick a number for D. Let’s choose “84” since it is a multiple of both 6 and 14. If $84 = 6\text{mph} \times T$, then we know that $T = 14$ hours. If $84 = 14\text{mph} \times T$, then we know that $T = 6$ hours. Now we can use another formula, the Average Rate formula, to find the average speed for the WHOLE trip. Average Rate = Total Distance / Total Time Using our Picked Number of 84, we know that the Total Distance traveled would be 168 miles. The Total Time is 14 hours + 6 hours = 20 hours. So the Average Rate = $168 \text{ miles} / 20 \text{ hours} = 8.4 \text{ mph}$. It doesn’t matter that Tracey didn’t “really” go 168 miles, or that we know she didn’t “really” go 20 hours. We Picked a Number just so that we could find the ratio of the Total Distance to the Total Time in order to calculate the Average Rate of the ENTIRE journey.

Now that we have found the Average Rate for the whole trip, we can plug it in to the “DIRT” formula to find the ACTUAL distance for the entire journey.

$D = R \times T$

$D = 8.4\text{mph} \times 1 \text{ hour}$

We know that $T = 1$ hour because the problem told us so. Therefore, the actual distance for the entire trip was 8.4 miles. The problem asks how many miles the trail was one way. $8.4 / 2 = 4.2$. The answer to the question is 4.2 miles.

Sample Questions:

25. Robert rides his scooter an average of 5 miles per hour, and Cindy rides her roller blades an average of 6 miles per hour. At these rates, how much longer does it take Runner A than Runner B to travel 3 miles?
26. A train traveling south travels 50 miles per hour for 3 hours; A second train traveling south travels 105 miles per hour for 2 hours. What is the difference, in miles between the number of miles traveled by the second bus and the number of miles traveled by the first bus?
27. A cattle train left Miami and traveled toward New York. 14 hours later a diesel train left traveling at 45 km/h in an effort to catch up to the train. After traveling for four hours, the diesel train finally caught up. What was the cattle train's average speed?
28. Jose left the White House and drove toward the recycling plant at an average speed of 40 km/h. Rob left some time later driving in the same direction at an average speed of 48 km/m. After driving for five hours, Rob caught up with Jose. How long did Jose drive before Rob caught up?
29. A cargo plane flew to the maintenance facility and back. It took one hour less time to get there than it did to get back. The average speed on the trip there was 220 mph. The average speed on the way back was 200 mph. How many hours did the trip there take?
30. Kali left school and traveled toward her friend's house at an average speed of 40 km/h. Matt left one hour later and traveled in the opposite direction with an average speed of 50 km/h. Find the number of hours Matt needs to travel before they are 400 km apart.
31. A submarine left Hawaii two hours before an aircraft carrier. The vessels traveled in opposite directions. The aircraft carrier traveled at 25 mph for nine hours. After this time the vessels were 280 mi. apart. Find the submarine's speed.

AVERAGE RATE: WORK RATE PROBLEMS

To work out these problems, it is usually assumed that a work-agent (say a man) takes certain number of time units T (usually days or hours) to complete the work. So the work rate of the agent in one time unit (a day or an hour) is expressed as $\frac{1}{T}$ th portion of the total amount of work.

The reason why this work rate in terms of work portion per unit time is the most important concept in Time and Work problems is - it makes possible summing up of efforts of more than one type of work agents working together at different work rates over unit time. This is the core concept behind the deductions of Time and Work problem of any type.

Example: A job can be completed by 4 men in 24 days and 4 women in 12 days. In how many days would the 4 men and 4 women working together complete the work?

Solution: In case of 4 men and 4 women working independently completing a job in 24 days and 12 days respectively, if we are asked to find the number of days taken to complete the job by 4 men and 4 women working together, by the conventional approach, we derive the per day work rate of 1 man and 1 woman as, $\frac{1}{4 \times 24} = \frac{1}{96}$ portion of work, and $\frac{1}{4 \times 12} = \frac{1}{48}$ portion of work.

When these two teams work together for 1 day, we would now be able to sum up their efforts in one day as, $\frac{4}{96} + \frac{4}{48} = \frac{3}{24} = \frac{1}{8}$ portion of work.

We now arrive at the desired result using unitary method. The number of days that the two teams would take to complete the job working together would just be inverse of the per day work portion, that is, 8 number of days.

This approach seems to be a bit complex as it deals with inverses, but this is the usual method followed.

Sample Questions:

32. It takes Trevon ten hours to clean an attic. Cody can clean the same attic in seven hours. Find how long it would take them if they worked together.

33. Working alone, Carlos can oil the lanes in a bowling alley in five hours. Jenny can oil the same lanes in nine hours. If they worked together how long would it take them?

34. Working together, Paul and Daniel can pick forty bushels of apples in 4.95 hours. Had he done it alone it would have taken Daniel 9 hours. Find how long it would take Paul to do it alone.

35. Working together, Jenny and Natalie can mop a warehouse in 5.14 hours. Had she done it alone it would have taken Natalie 12 hours. How long would it take Jenny to do it alone?

MEAN, MEDIAN, MODE, RANGE**Mean**

The "mean" is the "average", where you add up all the numbers and then divide by the number of numbers.

Median

The "median" is the "middle" value in the list of numbers, like the median strip on the highway. To find the median, your numbers have to be listed in numerical order, so you may have to rewrite your list first.

Mode

The "mode" is the value that occurs most often--mOde, mOst. If no number is repeated, then there is no mode for the list.

Range

The "range" is just the difference between the largest and smallest values.

Sample Questions:

36. What is the difference between the mean and the median of the set {2, 5, 18, 22} ?

37. To increase the mean of 4 numbers by 2, by how much would the sum of the 4 numbers have to increase?

38. The weekly salaries of six employees at McDonalds are \$140, \$220, \$90, \$180, \$140, \$200. For these six salaries, find: (a) the mean (b) the median (c) the mode and (d) the range.

Use the following information to answer the next three problems:

In January of 2006, your family moved to a tropical climate. For the year that followed, you recorded the number of rainy days that occurred each month. Your data contained 14, 14, 10, 12, 11, 13, 11, 11, 14, 10, 13, 8.

39. Find the mean, mode, median and range for your data set of rainy days.

40. If the number of rainy days doubles each month in the year 2007, what will be the mean, mode, median and range for the 2007 data?

41. If, instead, there are three more rainy days per month in the year 2007, what will be the mean, mode, median and range for the 2007 data?

COUNTING THE POSSIBILITIES

The fundamental counting principle: If there are ***m* ways** one event can happen and ***n* ways** a second event can happen, then there are ***m* × *n* ways** for the two events to happen. For example, with 7 shirts and 5 pairs of pants to choose from, you can put together $7 \times 5 = 35$ different outfits.

For more items, multiply all the ways together. For example, with 7 shirts, 5 pairs of pants, 8 ties and 2 pairs of shoes to choose from, you can put together $7 \times 5 \times 8 \times 2 = 560$ different outfits.

COMBINATIONS: THE SLOT METHOD

Combination problems ask you how many different ways a number of things could be chosen or combined. The rules for combination problems on the ACT are straightforward.

- Figure out the number of slots you need to fill.
- Fill in those slots.
- Find the product.

Example: At the school cafeteria, students can choose from 3 different salads, 5 different main dishes, and 2 different desserts. If Isabel chooses one salad, one main dish, and one dessert for lunch, how many different lunches could she choose?

- 10
- 15
- 25
- 30
- 50

Solution: There are three slots to fill, one for each item: salad, main dish, dessert. And the number of possibilities for each is pretty clear. Set up the slots and take the product as your answer.

$$\frac{3}{\text{Salad}} \times \frac{5}{\text{Main}} \times \frac{2}{\text{Dessert}} = 30 \quad \text{The correct answer here is choice (D).}$$

Sample Questions:

- In the school cafeteria, students choose their lunch from 4 sandwiches, 2 soups, 3 salads, and 6 drinks. How many different lunches are possible for a student who chooses exactly 1 sandwich, 1 soup, 1 salad, and 1 drink?
- Reggie knows how to make 5 different entrees, 4 different side dishes, and 6 different desserts. How many distinct complete meals, each consisting of an entrée, a side dish, and a dessert, can Reggie make?
- Chris owns 10 different dress shirts, 4 different pairs of pants, and 7 different ties. How many distinct outfits, each consisting of a shirt, a pair of pants, and a tie, can Justin make?

45. At summer camp, one boy and one girl will be selected to lead the weekly activities. If there are 130 boys and 145 girls at the camp, how many different 2-person combinations of 1 boy and 1 girl are possible?
46. Rana and Tom own a pizza shop, which offers 3 kinds of cheese, 4 kinds of meat toppings, and 5 kinds of vegetable toppings. Each type pizza on the menu has a combination of exactly 3 ingredients: 1 cheese, 1 meat, and 1 vegetable. How many types of pizzas are possible?
47. A hiking group will go from a certain town to a certain village by van on 1 of 3 roads, from the village to a waterfall by riding bicycles on 1 of 8 bicycle paths, and then from the waterfall to their campsite by hiking on 1 of 5 trails. How many routes are possible for the hiking group to go from the town to the village to the waterfall to their campsite?
48. The bowling league selects its 4 officers by first selecting the president, then the vice president, then the secretary, then the treasurer. If there are 40 bowlers who are eligible to hold office and no member can hold more than one office, which of the following gives the number of different results of the election?

Answer Key

1. $11/15$
2. $13/6$ or $2 \frac{1}{6}$
3. $5x/3$
4. $x - 2/3$
5. 385
6. 20
7. 25
8. $31/60$
9. \$2.53
10. 81
11. \$6.86
12. 89
13. 16
14. $a/2$
15. 90%
16. 46%
17. 80%
18. 35%
19. 4mg
20. 5 gal
21. 9.4 kg
22. 3.35
23. 3.175
24. Need to have 3.2 or greater, so B+ to be able to play
25. 0.1 hr longer
26. 60 mi/hr
27. 10 mi/hr
28. 1 hr
29. 10 hrs
30. 4 hrs
31. 5 mph
32. 4.12 hours
33. 3.21 hours
34. 11 hrs
35. 9 hrs
36. 5.25
37. 16
38.
 - a) Mean = $161.\bar{6}$
 - b) Median = 160
 - c) Mode = 140
 - d) Range = 130
39.
 - a) Mean = 9.75
 - b) Median = 11.5
 - c) Mode = 11 and 14
 - d) Range = 6
40.
 - a) Mean = 23.5
 - b) Median = 23
 - c) Mode = 22 and 28
 - d) Range = 12
41.
 - a) Mean = 14.75
 - b) Median = 14.5
 - c) Mode = 14 and 17
 - d) Range = 6
42. 144
43. 120
44. 280
45. 18,850
46. 60
47. 120
48. $40 \times 39 \times 38 \times 37$