Name:___________________________________ Date:_________________________

Math 2 Variable Manipulation Part 5
Quadratic Functions

MATH 1 REVIEW
THE CONCEPT OF FUNCTIONS
The concept of a function is both a different way of thinking about equations and a different way of notating equations. Let's look at a linear relationship the "old" way and with function notation.

Linear relationship: y = 2x + 1

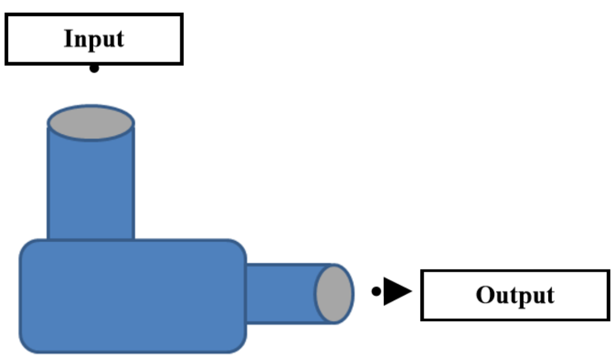
We say that x is the *independent variable* and y is the *dependent variable*. We choose values of x (the input) and then see what we get for y (the output). Another way to say this is: "The value of y is a function of (or depends on) our choice for x." When the above statement is written using mathematical symbols, it becomes:

$$y = f(x)$$

Read, "y equals f of x."

"y is a function of x"

One way to get a better grasp of the concept of a function is to picture the function as a machine. Our function machine is like a "**cause and effect map**" which typically shows the relationship between a cause and its effect. We will use a modified version to show the relationship between the input and output values in our "function machine." The domain values of x are the input (cause) and the range values of *f(x)* are the output (effect). Let's look at the function ***f(x) = -2x + 1***. When we input 0 for *x* (the cause), our output for *f(x)* is 1 (the effect).



Input x	$f(x) = -2x + 1$	Output $f(x)$
-2		
-1	$f(-1) = -2(-1) + 1$	3
0	$f(0) = -2(0) + 1$	1
1		
2		

Example: $f(x) = 2x^2 + 1$
Solution:

We pick random values for x, substitute them into the equation for x, and simplify. The numbers that we pick are the input values. Let's pick $f(-1)$, $f(0)$ and $f(2)$. We will use the same function each time.	$f(x) = 2x^2 + 1$ **Original Equation** $f(0) = 2(0)^2 + 1$ **Substitute (0) for x** $f(0) = 1$ **Solve**
$f(x) = 2x^2 + 1$ **Original Equation** $f(-1) = 2(-1)^2 + 1$ **Substitute (-1) for x** $f(-1) = 3$ **Solve**	$f(x) = 2x^2 + 1$ **Original Equation** $f(2) = 2(2)^2 + 1$ **Substitute (2) for x** $f(2) = 9$ **Solve**

DEPENDENT AND INDEPENDENT VARIABLES

A variable that depends on one or more other variables. For equations such as y = 3x – 2, the dependent variable is y. The value of y depends on the value chosen for x. Usually the dependent variable is isolated on one side of an equation. Formally, a dependent variable is a variable in an expression, equation, or function that has its value determined by the choice of value(s) of other variable(s). The **in**dependent variable is the **in**put and the dependent variable is the output in the function scenario. The "in" is "in."

WRITING FUNCTIONS FROM DATA

In each of the examples above, the solutions were found from the function. However, functions can be derived from data as well. Remember the slope-intercept formula from coordinate geometry? This equation for linear lines works for linear functions as well.

$$y = mx + b$$

where m is the slope and b is the y intercept. First, use two points to calculate the slope or the rate of change of the function. Slope is the rise over the run or m = $\frac{y_2-y_1}{x_2-x_1}$. Then substitute one of the points into the equation for x and y and solve for b. See coordinate geometry for more practice. We can write the resulting equation as:

$$f(x) = mx + b$$

Example: Write an equation from the following data

Input	Output
3	7
5	11

Solution A: This is the same thing as finding the equation of a line through the points (3,7) and (5,11). First, calculate the slope from 2 points:

$$slope = \frac{y_2 - y_1}{x_2 - x_1} = \frac{11-7}{5-3} = \frac{4}{2} = 2$$

Then substitute the slope and either point into the slope-intercept formula and solve for the y-intercept (b).

$$\begin{aligned} 7 &= 2(3) + b \\ 7 &= 6 + b \\ -6 &\quad -6 \\ \hline 1 &= b \end{aligned}$$

Then substitute the slope and y-intercept into the slope-intercept formula to get

y = 2x + 1

We can also write an equation using the Point-Slope Formula:

$$y - y_1 = m(x - x_1)$$

where m is the slope and (x_1, y_1) is a point on the line. This formula is sometimes more useful than the slope-intercept form in that you do not need the y-intercept, just a point on the line and the slope to create an equation of the line.

Solution B: Find the slope as above, but then use one of the data points and input into the point-slope formula.

using (3,7):	or	using (5,11):
y − 7 = 2(x− 3)		y − 11 = 2(x − 5)

Each of these three equations are valid equations for the data set.

MATH 2 LEVEL
FUNCIONS THAT ARE QUADRATIC IN NATURE

Math 2 functions build on the basic concepts learned in Math 1 with increased complexity. Some examples are functions with multiple variables and quadratic functions. Quadratic functions can be manipulated in similar ways as linear functions. The graphs of quadratic functions are parabolic in nature.

Example: Let a function of 2 variables be defined by $f(a, b) = ab - (a - b)$. What is the value of $f(8, 9)$?
Solution: The question asks for $f(8, 9)$ which means that a will = 8 and b will = 9. So just plug in both of these values for a and b.
$f(a, b) = ab - (a - b)$
$f(8, 9) = 8(9) - (8 - 9) = 72 - (-1) = 71$

Example: If $f(x) = x^2 + 3$, then $f(x + y) = ?$
Solution: For every x in f(x), plug in x + y and then simplify
$f(x + y) = (x + y)^2 + 3$
$= (x + y)(x + y) + 3$
$= x^2 + 2xy + y^2 + 3$

Example: Let the function g be defined by $g(x) = 3(x^2 - 2)$. When $g(x) = 69$, what is a possible value of 2x -3?
Solution: Since the question gives a solution for $g(x)$, first solve the function for x and then plug that answer into $2x - 3$.
$3(x^2 - 2) = 69$
$3x^2 - 6 = 69$
$3x^2 = 75$
$x^2 = 25$
$x = 5$
Plugging 5 into the equation: $2x - 3 = 2(5) - 3 = 7$

Sample Questions:

1. If $f(a) = a^2 - 2$, then $f(a + b) = ?$

2. Find $R(w) = w^3 + 3w^2 - 5w - 6$ when $w = -2$

3. Find $f(3h + 2)$ when $f(x) = x^2 + 2x - 1$

4. The table below gives values of the quadratic function f for selected values of x. Which of the following defines the quadratic function f?

x	0	1	2	3
F(x)	-6	-5	-2	3

a. $f(x) = x^2 - 6$
b. $f(x) = x^2 + 6$
c. $f(x) = 2x^2 - 10$
d. $f(x) = 2x^2 - 6$
e. $f(x) = 2x^2 - 7$

5. In the xy-coordinate system, $(\sqrt{5}, s)$ is one of the points of intersection of the graphs $y = 2x^2 + 6$ and $y = -4x^2 + m$, where m is a constant. What is the value of m?

6. Given $f(x) = 3x + 5$ and $g(x) = x^2 - x + 7$, which of the following is an expression for $f(g(x))$?
   a. $3x^2 - 3x + 26$
   b. $3x^2 - 3x + 12$
   c. $x^2 - x + 12$
   d. $9x^2 + 25x + 27$
   e. $3x^2 + 21$

7. Given $f(x) = x - \frac{1}{x}$ and $g(x) = \frac{1}{x}$, what is $f(g(\frac{1}{2}))$?

FORMS OF QUADRATIC FUNCIONS
Standard Form: $f(x) = ax^2 + bx + c$, where $a \neq 0$
Example: $f(x) = 4x^2 - 6x + 3$

Vertex Form: $f(x) = a(x - h)^2 + k$, where $a \neq 0$
Example: $f(x) = 2(x + 3)^2 + 5$

Factored Form: $f(x) = a(x - p)(x - q)$, where $a \neq 0$
Example: $f(x) = 2(x - 4)(x - 7)$

Zero Function: is a value of the input x that makes the output *f(x)* equal to zero. The zeros of a function are also known as roots, x-intercepts, and solutions of $ax^2 + bx + c = 0$

Zero Product Property: states that if a product of two quantities equals zero, at least one of the quantities equals zero.
Example: If ab = 0 then a = 0 or b = 0 or both a and b = 0

FINDING ZEROS (INTERCEPTS) IN A QUADRATIC FUNCTION
When a function is in factored form, the Zero Product Property can be used to find the zeros of the function. If $f(x) = ax(x - p)$ then $ax(x - p) = 0$ can be used to find the zeros of *f(x).* If $0 = ax(x - p)$ then either $ax = 0$ or $(x - p) = 0$. Therefore, x = 0 or x = p
Example: Find the zeros of $f(x) = 2x(x + 7)$
Solution: $f(x) = 2x(x + 7)$ so $2x(x + 7) = 0$
Set each factor = 0 so 2x = 0 or x + 7 = 0
x = 0 or x -7
The zeros are (0.0) and (-7,0)

If $f(x) = (x - p)(x - q)$ then $(x - p)(x - q) = 0$ can be used to find the zeros of *f(x).* If $(x - p)(x - q) = 0$
Then either $(x - p) = 0$ or $(x - q) = 0$. Therefore, either x = p or x = q.
Example: $f(x) = (x - 5)(x + 9)$
Solution: $f(x) = (x - 5)(x + 9)$ so $(x - 5)(x + 9) = 0$
x – 5 = 0 or x + 9 = 0
x = 5 or x = -9
The zeros are (5.0 and (-9,0)

If a quadratic function is given in standard form, factor first then apply the Zero Product Property.

Example: $f(x) = x^2 - 11x + 24$
Solution: $f(x) = x^2 - 11x + 24$ so $x^2 - 11x + 24 = 0$
Factor to (x – 8)(x – 3) = 0
x-8 = 0 or x – 3 = 0
x = 8 or x = 3
The zeros are (8,0 and (3,0)

Sample Questions:

8. Find the zeros of $f(x) = 4x^2 - 4x - 15$

9. Find the zeros of $f(x) = -x(x + 7)$

10. Find the zeros of $f(x) = 2x(x - 6)$

11. Find the zeros of $f(x) = (x - 21)(x - 3)$

12. Find the zeros of $f(x) = x^2 - 7x + 6$

13. Find the zeros of $f(x) = 9x^2 - 25$

14. Find the zeros of $f(x) = 5x^2 - 4x - 12$

FINDING MAXIMUM/MINIMUM (THE VERTEX) OF A QUADRATIC FUNCTION

Remember when a quadratic function is in the vertex form $f(x) = a(x – h)^2 + k$ the point (h,k) is the vertex of the parabola. The value of a determines whether the parabola opens up or down. The vertex of a parabola that opens us, when $a > 0$, it is the minimum point of a quadratic function. The vertex of a parabola that opens up, when $a < 0$, is the maximum point of a quadratic function.

Example: Find the vertex of $f(x) = (x + 2)^2 – 3$, then determine whether it is a maximum or minimum point.
Solution: $f(x) = (x + 2)^2 – 3$ $\quad$ $f(x) – (x – (-2))^2 + (–3)$
So the vertex is (-2, -3) and since the leading coefficient is 1, which makes $a > 0$, the vertex is a minimum.

Sample Questions:

15. Find the vertex of $f(x) = (x - 8)^2 + 4$

16. Find the vertex of $f(x) = 4(x - 5)^2 – 3$

17. Find the vertex of $f(x) = -(x + 3)^2 + 7$

18. Find the vertex of $f(x) = 6x^2 + 5$

COMPLETING THE SQUARE

If a quadratic function is given in standard form, complete the square to rewrite the equation in vertex form. To complete the square of $x^2 \pm bx$, add $(\frac{b}{2})^2$. In other words, divide the x coefficient by two and square the result. Solution as follows:

$x^2 + bx +$ _____
$x^2 + bx + (\frac{b}{2})^2$
$(x + \frac{b}{2})(x + \frac{b}{2})$
$(x + \frac{b}{2})^2$

Example: Complete the square of $x^2 + 6x$
Solution: $x^2 + 6x +$ _____
$x^2 + 6x + (\frac{6}{2})^2$
$x^2 + 6x + (3)^2$
$x^2 + 6x + 9$
$(x + 3)(x + 3)$ OR $(x + 3)^2$

To complete the square of $ax^2 \pm bx$, factor out the leading coefficient, a, giving $(x^2 + \frac{b}{a}x)$. Now add add $(\frac{b}{2a})^2$, which is the square of the coefficient of x divided by two. Solution as follows:

$ax^2 + bx +$ _____
$a(x^2 + \frac{b}{a}x +$ _____$)$
$a(x^2 + \frac{b}{a}x + (\frac{b}{2a})^2)$
$a(x + \frac{b}{2a})(x + \frac{b}{2a})$
$a(x + \frac{b}{2a})^2$

Example: Complete the square of $3x^2 - 6x$

Solution: $3x^2 + 6x +$ ____

$3(x^2 - \frac{6}{3}x + \_\_\_)$

$3(x^2 - \frac{6}{3}x + (\frac{6}{2*3})^2)$

$3(x^2 - 2x + (1)^2)$

$3(x - 1)(x - 1)$

$3(x - 1)^2$

Sample Questions:

19. $x^2 + 10x$

20. $x^2 - 14x$

21. $x^2 - 22x$

22. $x^2 - 38x$

23. $m^2 + 3m$

24. $n^2 + \frac{1}{2}n$

25. x2 − 34x

COMPLETING THE SQUARE TO FIND VERTEX, MINIMUM AND MAXIMUM

If a quadratic function is given in standard form, complete the square to rewrite the equation in vertex form. To find the vertex of f(x) = x^2 + bx + 7 and determine if it is a maximum or minimum, first complete the square, of (x^2 + bx) and keep +7 outside the parenthesis. Then subtract $(\frac{b}{2})^2$ outside the parenthesis to maintain equality. In other words, you are really adding zero to the equation. Simplify and combine like terms to solve for h and k and determine if the leading coefficient is $>$ or $<$ 0 to determine vertex, and maximum or minimum.

Example: $f(x) = x^2 + 12x + 7$

Solution: $f(x) = (x^2 + 12x + \_\_\_) + 7$

$f(x) = (x^2 + 12x + (\frac{12}{2})^2) + 7 - (\frac{12}{2})^2$

$f(x) = (x^2 + 12x + 6^2) + 7 - 6^2$

$f(x) = (x^2 + 12x + 36) + 7 - 36$

$f(x) = (x + 6)^2 - 29$

Vertex = (-6, -29) and since the leading coefficient is 1 or a $>$ 0, the vertex is a minimum

Example: Find the vertex of $f(x) = 3x^2 + 18x - 2$

Solution: $f(x) = (3x^2 + 18x + \_\_\_\_) - 2$

$f(x) = 3(x^2 + 6x + \_\_\_) - 2$

$f(x) = 3(x^2\ \ 6x + (\frac{6}{2})^2) - 2 - 3(\frac{6}{2})^2$

$f(x) = 3(x^2 + 6x + (3)^2) - 2 - 3*3^2$

$f(x) = 3(x^2 + 6x + 9) - 2 - 27$

$f(x) = 3(x + 3)^2 - 29$

h = -3 and k = -29 and since the leading coefficient is 3 or a $>$ 0, the vertex is a minimum

Example: Find the vertex of $f(x) = -4x^2 - 8x + 3$

Solution: $f(x) = (-4x^2 - 8x + \_\_\_\_) + 3$

$f(x) = -4(x^2 + 2x + \_\_\_) + 3$

$f(x) = -4 (x^2 + 2x + (\frac{2}{2})^2) + 3 - (-4)(\frac{2}{2})^2$

$f(x) = -4 (x^2 + 2x + (1)^2) + 3 - (-4)*1^2$

$f(x) = -4 (x^2 + 2x + 1) + 3 + 4$

$f(x) = -4 (x + 1)^2 + 7$

Vertex is (-1, 7) and since the leading coefficient is -4 or a $<$ 0, the vertex is a maximum

Sample Questions:

26. Find the vertex of $f(x) = x^2 + 10x - 20$ by completing the square and determine if the vertex is a maximum or minimum

27. Find the vertex of $f(x) = 5x^2 - 20x - 9$ by completing the square and determine if the vertex is a maximum or minimum

28. Find the vertex of $f(x) = x^2 + 8x + 10$ by completing the square and determine if the vertex is a maximum or minimum

WORD PROBLEMS WITH VERTEX, INTERCEPTS, MINIMUM AND MAXIMUM

Various word problems can be modeled by quadratic equations. These will most often be written in standard form, so complete the square to rewrite the equation in vertex form. Then find the vertex and maximum or minimum. You can also find where x or y = 0.

Sample Questions:

29. The height h(t), in feet, of a "weeping willow" firework display, t seconds after having been launched from an 80-ft high rooftop, is given by $h(t) = -16t^2 + 64t + 80$. When will it reach its maximum height? What is its maximum height?

30. The value of some stock can be represented by $V(x) = 2x^2 - 8x + 10$, where x is the number of months after January 2012. What is the lowest value V(x) will reach, and when did that occur?

31. A projectile is launched at a speed of 40 meters per second from a 89-meter tall platform. The equation for the object's height *S* at time *t* seconds after launch is *S(t) = -4.9t2 + 40t + 89*, where S is in meters. How long will it take for the object to reach it's maximum height?

Answer Key

1. $a^2 + 2ab + b^2 - 2$
2. R(-2) = 8
3. $9h^2 + 18h + 7$
4. $f(x) = x^2 – 6$ (A)
5. 36
6. $3x^2 - 3x + 26$ (A)
7. 3/2
8. (5/2, 0)(-3/2, 0)
9. (0, 0)(-7, 0)
10. (6, 0)(0, 0)
11. (21, 0)(3, 0)
12. (6, 0)(1, 0)
13. (5/3, 0)(-5/3, 0)
14. (-6/5, 0)(2, 0)
15. (8, 4) minimum
16. (5, -3) minimum
17. (-3, 7) maximum
18. (0, 5) minimum
19. $x^2 + 10x + 25$
20. $x^2 – 14x + 49$
21. $x^2 - 22x + 121$
22. 361
23. 9/4
24. 1/16
25. 289
26. (-5, -45), minimum
27. (2, -29), minimum
28. (-4, -6) minimum
29. 2 seconds, 144 feet
30. In March 2012, it hits a low value of 2
31. 4 seconds