Math 3 Variable Manipulation Part 1
Algebraic Systems

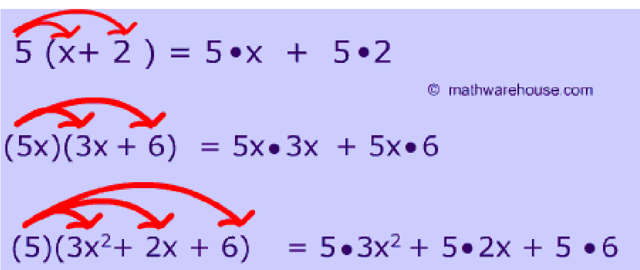
PRE ALGEBRA REVIEW OF INTEGERS (NEGATIVE NUMBERS)

Concept	Example
Adding positive numbers is just simple addition	2 + 3 = 5
Subtracting positive numbers is just simple subtraction	6 − 3 = 3
Subtracting a Negative is the same as Adding	6−(−3) = 6 + 3 = 9
Subtracting a Positive *or* Adding a Negative *is* Subtraction	5+(−2) = 5 **−** 2 = 3 −6+(+3) = −6 **+** 3 = -3
When you multiply a negative number by a positive number then the product is always negative	3·(−4) = −12
When you divide a negative number by a positive number then the quotient is negative. OR when you divide a positive number by a negative number then the quotient is also negative	$\frac{-12}{3}$ = -4 $\frac{12}{-3}$ = -4
When you multiply two negative numbers or two positive numbers then the product is always positive	(−3)·(−4) = 12
When you divide two negative numbers then the quotient is positive	$\frac{-12}{-3}$ = 4

MATH 1 REVIEW

DISTRIBUTIVE PROPERTY

The distributive property is one of the most frequently used properties in math. It lets you multiply a sum by multiplying each addend separately and then add the products.



ADDING AND SUBTRACTING MONOMIALS {COMBINING LIKE TERMS)

Combining like terms is just a way to count the number of items that are the same. Another way is to keep the variable part unchanged while adding or subtracting the coefficients (numbers):

Example: 2a + 3a

Solution 2: Add the coefficients and keep the variables the same. 2a + 3a = (2 + 3)a = 5a

MULTI-STEP EQUATIONS

The steps to solving a multi-step equation are:

1. Distribute
2. Collect like terms
3. Gather all the variables on one side
4. Gather all the constants on other side
5. The variable should have a coefficient of 1
6. Check your answer

LIST OF ALGEBRAIC PROPERTIES

Property (a, b and c are real numbers)	**Example**
Identity Property of Addition Any number when added to 0 is itself.	$c+0=c$
Identity Property of Multiplication Any number multiplied by 1 is itself.	$b\cdot 1=b$
Multiplicative Property of Zero Any number multiplied by zero is zero.	$a\cdot 0=0$
Commutative Property of Addition or Multiplication The order in which you add or multiply does not change the sum or product.	$a+b=b+a$ $ab=ba$
Associative Property of Addition or Multiplication The order in which you group numbers does not change the sum or product.	$(a+b)+c=a+(b+c)$ $(ab)c=a(bc)$
Symmetric Property If one number is equal to a second number, then the second number is equal to the first.	If $a=b$ then $b=a$
Substitution Property If $a=b$ then a can be substituted into an equation for b.	If $a=b$ and $b+c=6$ then $a+c=6$.
Distributive Property The product of a number and a sum or difference is the same as the sum or difference of the products of the number and each element of the sum or difference.	$a(b+c)=ab+ac$ $a(b-c)=ab-ac$
Additive Inverse The sum of a number and its inverse is zero.	$a+(-a)=0$
Multiplicative Inverse The product of a number and its inverse is one.	$b\cdot\frac{1}{b}=1$
Addition Property of Equality Adding the same number to each side of an equation produces an equivalent equation.	If $a=b$ then $a+c=b+c$
Subtraction Property of Equality Subtracting the same number from each side of an equation produces an equivalent equation.	If $a=b$ then $a-c=b-c$
Multiplication Property of Equality Multiplying each side of the equation by the same number produces an equivalent equation.	If $a=b$ then $ac=bc$
Division Property of Equality Dividing each side of the equation by the same number produces an equivalent equation.	If $a=b$ then $\frac{a}{c}=\frac{b}{c}$

SOLVING A LINEAR EQUATIONS (SOLVE FOR X)

To solve an equation, isolate the variable. As long as you do the same thing to both sides of the equation, the equation is still balanced.

Example: 5x - 12 = -2x + 9

Solution: First get all the x terms on one side by adding 2x to both sides: 7x - 12 = 9. Then add 12 to both sides: 7x = 21, then divide both sides by 7 to get: x = 3.

Sample Questions:

1. -10(-9x + 9) - 8x = 9 + 3x – 4(2x + 3)

2. $-\frac{4}{7}=\frac{2x+5}{x-8}$

3. $\frac{2}{3} = -\frac{3}{5}t$

4. 9(7x + 4) – 3(2x – 5) = 3x – 4(x + 2) + x(3 – 5) - 1

5. $\frac{3}{4}(2x+1) = 2$

6. $\frac{1}{3} + 2m = m - \frac{3}{2}$

7. -6(m – 4) = -10m + 3(8 + 8m)

8. j(h – 9) + 2; use h = 9 and j = 8

9. 15(2x) + 15 – 3 = 16 + 30x

10. -8(-5x – 6) + 7(1 + 9x) = 53 + 11x(9)

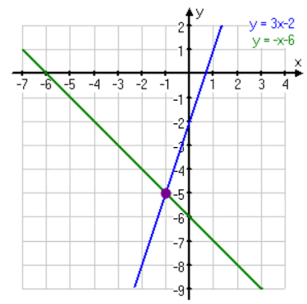
11. If 8y = 3x - 11, then x =
    a. (88/3) y
    b. (8/3) y + 11
    c. (8/3) y – 11
    d. (8y - 11)/3
    e. (8y + 11)/3

MATH 2 REVIEW

SYSTEM OF EQUATIONS INTRODUCTION

A "system" of equations is a set or collection of equations that you deal with all together at once. Consider the following two-variable system of linear equations:

$$y = 3x - 2 \text{ and } y = -x - 6$$

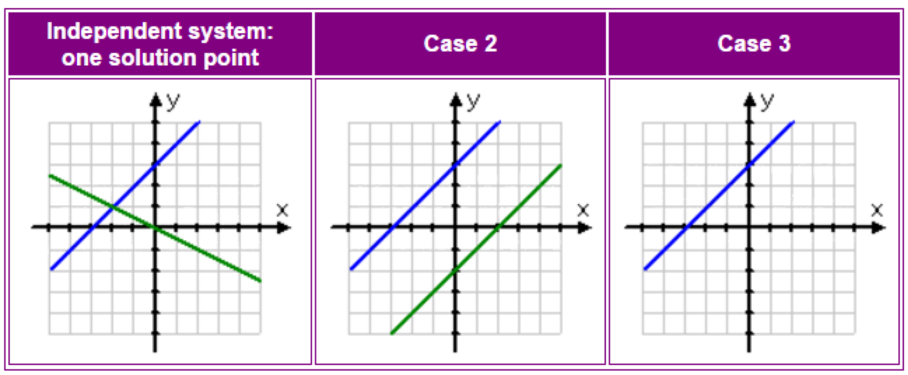


Since the two equations above are in a system, we deal with them together at the same time. The purple point is a solution to the system, because it lies on both of the lines. Since this point is on both lines, it thus solves both equations, so it solves the entire system of equation. This relationship is always true: For systems of equations, "solutions" are "intersections". You can confirm the solution by plugging it into the system of equations, and confirming that the solution works in each equation.

Note: You can solve for two variables only if you have two distinct equations. If you have one variable, you only need one equation, but if you have two variables, you need two distinct equations. Two forms of the same equation will not be adequate. If you have three variables, you need three distinct equations, and so on.

SOLVING A SYSTEM OF EQUATIONS BY GRAPHING

When graphically finding intersections of lines. For two-variable systems, there are then three possible types of solutions:



The first graph above, "Case 1", shows two distinct non-parallel lines that cross at exactly one point. This is called an "independent" system of equations, and the solution is always some *x*,*y*-point.

The second graph above, "Case 2", shows two distinct lines that are parallel. Since parallel lines never cross, then there can be no intersection; that is, for a system of equations that graphs as parallel lines, there can be no solution. This is called an "inconsistent" system of equations, and it has no solution.

The third graph above, "Case 3", appears to show only one line. Actually, it's the same line drawn twice. These "two" lines, really being the same line, "intersect" at every point along their length. This is called a "dependent" system, and the "solution" is the whole line.

This shows that a system of equations may have one solution (a specific *x*,*y*-point), no solution at all, or an infinite solution (being all the solutions to the equation). You will never have a system with two or three solutions; it will always be one, none, or infinitely-many.

Solving systems of equations "by graphing," is a bit tricky because the only way you can find the solution from the graph is *IF* you draw a very neat axis system, *IF* you draw very neat lines, *IF* the solution happens to be a point with nice neat whole-number coordinates, and *IF* the lines are not close to being parallel.

SOLVING A SYSTEM OF EQUATIONS BY SUBSTITUTION

The method of solving "by substitution" works by solving one of the equations (you choose which one) for one of the variables (you choose which one), and then plugging this back into the other equation, "substituting" for the chosen variable (so that you only have one equation with one variable) and solving for the other. Then you back-solve for the first variable.

Example: Solve the following system by substitution.

$2x - 3y = -2$
$4x + y = 24$

Solution: The idea here is to solve one of the equations for one of the variables, and plug this into the other equation. It does not matter which equation or which variable you pick. There is no right or wrong choice; the answer will be the same, regardless. But — some choices may be better than others.

For instance, in this case, it would probably be simplest to solve the second equation for "$y =$", since there is already a y by itself in the equation. Solving the first equation for either variable, would yield fractions, and solving the second equation for x would also give fractions. It wouldn't be "wrong" to make a different choice, but it would probably be more difficult. So, solve the second equation for y:

$4x + y = 24$
$y = -4x + 24$

Now plug this in ("substitute it") for "y" in the first equation, and solve for x:

$2x - 3(-4x + 24) = -2$
$2x + 12x - 72 = -2$
$14x = 70$
$x = 5$

Now, plug this x-value back into either equation, and solve for y. But since there is already an expression for "$y =$", it will be simplest to just plug into this:

$y = -4(5) + 24 = -20 + 24 = 4$ Then the solution is $(x, y) = (5, 4)$.

Warning: If I had substituted "$-4x + 24$" expression into the same equation used to solve for "$y =$", the answer would be 24 = 24. This does not help solve for x. Make sure to substitute into the *other* equation.

Note: If you correctly solve a system of equations and get something obviously wrong line -12 = 3, then the answer is “No Solution”. This means that the lines do not intersect—there is no solution to the system of equations. If you correctly solve a system of equations and get something like 4 = 4, you know that both equations are actually the same equation—the same line, they are not independent from each other, so every point on the line is a solution.

SOLVING A SYSTEM OF EQUATIONS BY ELIMINATION

The method of elimination for solving systems of equations is also called the addition method. You add the equations together so that one of the variables will cancel out and you only have one equation with one variable which is easily solvable and then you can substitute the answer into one of the equations and solve for the second variable.

Example: Solve the following system using the elimination method.

$2x + y = 9$
$3x - y = 16$

Solution: Note that, when the equations are added, the y's will cancel out. So draw an "equals" bar under the system, and add down:

$2x + y = 9$
$\underline{3x - y = 16}$
$5x \quad = 25$

Now I can divide through to solve for $x = 5$, and then back-solve, using either of the original equations, to find the value of y. The first equation has smaller numbers, so I'll back-solve in that one:

$2(5) + y = 9$
$10 + y = 9$
$y = -1$ Then the solution is $(x, y) = (5, -1)$.

It doesn't matter which equation you use for the back solving; you'll get the same answer either way. If I'd used the second equation, I'd have gotten:

$3(5) - y = 16$
$15 - y = 16$
$-y = 1$ or $y = -1$...which is the same result as before.

Some equations are not set up as easily as the previous example. You may need to modify one or both of the equations to make them into two equations that are easily added to cancel out one of the variables. One option is to multiply one equation by negative 1 or another number that will when added to the other equation will cancel out one of the variables so that you only have one equation with one variable. The main strategy is to combine the equations in such a way that one of the variables cancels out.

Example: Solve the following system using elimination method

4x + 3y = 8
x + y = 3

Solution: Multiply both sides of the second equation by -3 to get: -3x - 3y = -9. Now add the equations; the 3y and the -3y cancel out, leaving x = -1:

$4x + \cancel{3y} = 8$
$\underline{+(-3x - \cancel{3y} = -9)}$
$x \quad = -1$ Plug into one of the original equations and you'll find that y = 4.

Then the solution is $(x, y) = (-1, 4)$

Example: Solve the following system using elimination method

$4x - 3y = 25$
$-3x + 8y = 10$

Solution: Since nothing easily cancels, there is not a number than you can multiply one equation by to cancel, both equations must be multiplied to create a cancellation. One option is to multiply to convert the x-terms to 12x's or the y-terms to 24y's. Since the 12 is smaller than 24, multiply the first equation by 3 and the second equation by 4 to cancel the x-terms. (You would get the same answer in the end if you set up the y-terms to cancel. They are both the "right way"; it was just a choice. You could make a different choice, and that would be just as correct.)

$$\begin{array}{rcr} 4x - 3y = 25 & \xrightarrow{3R_1} & 12x - 9y = 75 \\ -3x + 8y = 10 & \xrightarrow{4R_2} & \underline{-12x + 32y = 40} \\ & & 23y = 115 \end{array}$$

Solving, I get that $y = 5$. Using the first equation for back-solving:

$4x - 3(5) = 25$
$4x - 15 = 25$
$4x = 40$
$x = 10$ The solution is $(x, y) = (10, 5)$

Sample Questions:
Solve each system by substitution:

12. $3x - 5y = 17$
$y = -7$

13. $-7x + 2y = 18$
$6x + 6y = 0$

Solve each system by elimination:

14. $x - y = 11$
$2x + y = 19$

15. $-7x + y = -19$
$-2x + 3y = -19$

16. $16x - 10y = 10$
$-8x - 6y = 6$

17. $-x - 7y = 14$
$-4x - 14y = 28$

18. $-16y = 22 + 6x$
$-11y - 4x = 15$

SYSTEM OF EQUATIONS WORD PROBLEMS

There are many word problems that include system of equations. Translate the words into equations and use various system of equations methods to solve. Sometimes there may be three variables, but you can combine in such a way to cancel out two at a time and still get a solution to the problem.

Sample Questions:

19. The difference of two number is 3 and their sum is 13. Find the numbers.

20. Two small pitchers and one large pitcher can hold 8 cups of water. One large pitcher minus one small pitcher constitutes 2 cups of water. How many cups of water can each pitcher hold?

21. Margie is responsible for buying a week's supply of food and medication for the dogs and cats at a local shelter. The food and medication for each dog costs twice as much as those supplies for a cat. She needs to feed 164 cats and 24 dogs. Her budget is $4240. How much can Margie spend on each dog for food and medication?

22. The school that Stefan goes to is selling tickets to a choral performance. On the first day of ticket sales the school sold 3 senior citizen tickets and 1 child ticket for a total of $38. The school took in $52 on the second day by selling 3 senior citizen tickets and 2 child tickets. Find the price of a senior citizen ticket and the price of a child ticket.

23. A boat traveled 210 miles downstream and back. The trip downstream took 10 hours. The trip back took 70 hours. What is the speed of the boat in still water? What is the speed of the current?

24. The senior classes at High School A and High School B planned separate trips to New York City. The senior class at High School A rented and filled 1 van and 6 buses with 372 students. High School B rented and filled 4 vans and 12 buses with 780 students. Each van and each bus carried the same number of students. How many students can a van carry? How many students can a bus carry?

MATH 3 LEVEL:
SYSTEM OF THREE EQUATIONS AND THREE UNKONWNS

Similar to solving two variable equations, when solving for three variables, we express one variable in terms of another and substitute until we obtain a single equation with only one variable. The final equation tends to be rather large and at times complicated making substitution method not a very ideal method for solving three variable systems of equations. However, substitution method's simplicity overshadows all the complications and makes the method a very fundamental method for solving systems of equations. Systems with three equations and three variables can be solved using also be solved using the Addition/Subtraction method.

Steps to solve a system with three equations and three variables by elimination method:

1. Pick any two pairs of equations from the system.
2. Eliminate the same variable from each pair using the Addition/Elimination method.
3. Solve the system of the two **new** equations using the Addition/Elimination method.
4. Substitute the solution back into one of the original equations and solve for the third variable.
5. Check by plugging the solution into one of the other three equations.

Example: Solve the following system:

$4x - 3y + z = - 10$
$2x + y + 3z = 0$
$- x + 2y - 5z = 17$

Solution: Pick two pairs and eliminate the same variable from each system:

$4x - 3y + z = - 10$
$2x + y + 3z = 0$

$4x - 3y + z = - 10$
$-4x - 2y - 6z = 0$

$-5y - 5z = - 10$

$2x + y + 3z = 0$
$- x + 2y - 5z = 17$

$2x + y + 3z = 0$
$-2x + 4y - 10z = 34$

$5y - 7z = 34$

Solve the system of the two new equations:

$-5y - 5z = - 10$
$5y - 7z = 34$

$-12z = 24$ Thus, $z = - 2$

$-5y - 5(- 2) = - 10$
$-5y = - 20$ Thus, $y = 4$

Substitute into one of the original equations:

$- x + 2y - 5z = 17$
$- x + 2(4) - 5(- 2) = 17$
$- x + 18 = 17$
$- x = - 1 \quad x = 1$

Therefore, $(x, y, z) = (1, 4, - 2)$

Check: Does $2(1) + 4 + 3(- 2) = 0$? Yes

Sample Questions:

Solve by Substitution:

25. $y = x + z + 5$
    $z = -3y - 3$
    $2x - y = -4$

26. $y = x + 4z - 5$
    $4x + 3y - 2z = 5$
    $z = -2x + 2$

27. $3x - 3y = -6$
    $z = -3x - 3y + 9$
    $-4x + 5y + z = 8$

Solve by Elimination:

28. $4x + 4y + z = 24$
    $2x - 4y + z = 0$
    $5x - 4y - 5z = 12$

29. $x - 6y + 4z = -12$
    $x + y - 4z = 12$
    $2x + 2y + 5z = -15$

30. $5x - 4y + 2z = 21$
    $-x - 5y + 6z = -24$
    $-x - 4y + 5z = -21$

31. $5a + 5b + 5c = -20$
    $4a + 3b + 3c = -6$
    $-4a + 3b + 3c = 9$

Answers

1. 1
2. -1/6
3. -10/9 or -1 1/9
4. -1
5. 5/6
6. -11/6 or – 1 5/6
7. 0
8. 2
9. No solution
10. -1/2
11. E
12.
13. (−6, −7)
14. (−2, 2)
15. (10, −1)
16. (2, −5)
17. (0, −1)
18. (0, −2)
19. (−1, −1)
20. 2 and 4
21. $40 for each dog
22. senior citizen ticket: $8, child ticket: $14
23. boat: 12 mph, current: 9 mph
24. Van: 18, Bus: 59
25. (−2, 0, −3)
26. (0, 3, 2)
27. (1, 3, −3)
28. (4, 2, 0)
29. (0, 0, −3)
30. (5, −1, −4)
31. No unique solution