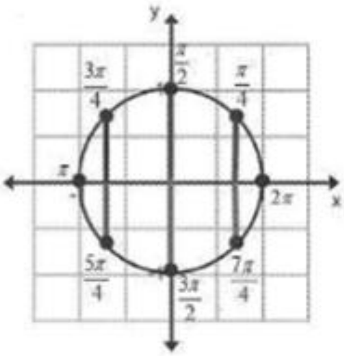
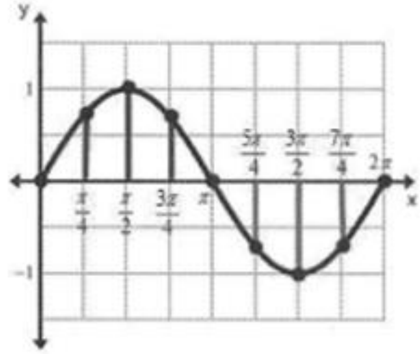
Math 3 Trigonometry Part 2
Waves & Laws

GRAPHING SINE AND COSINE

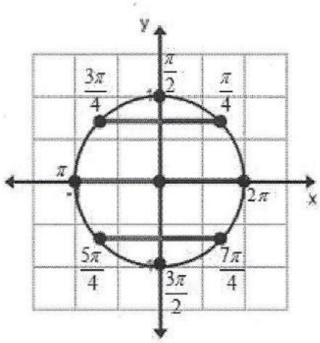
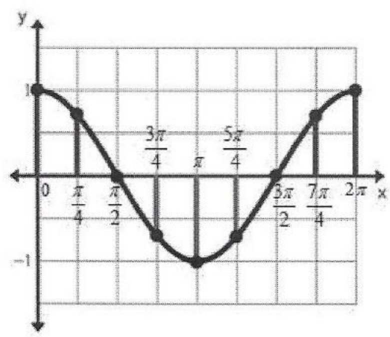
Graph of sine function:

Plotting every angle and its corresponding sine value, which is the y-coordinate, for different angles on the unit circle, allows us to create the sine function where x = θ and y = sinθ.

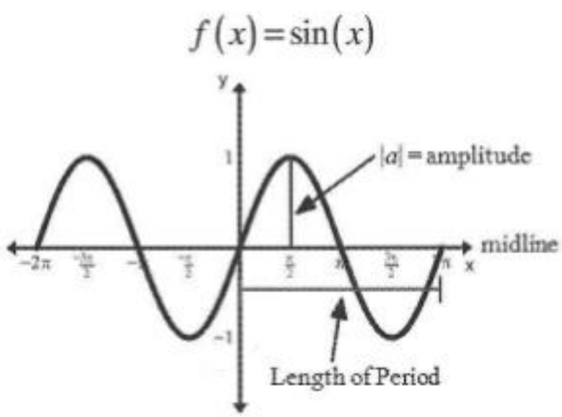



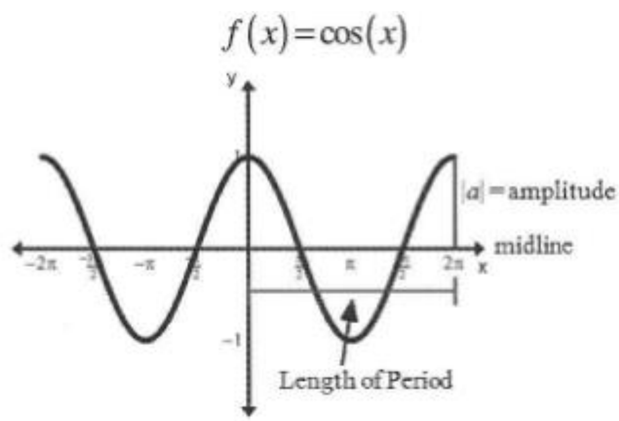
Graph of cosine function:

Plotting every angle and its corresponding cosine value, which is the x-coordinate, for different angles on the unit circle, alows us to create the cosine function where x = θ and y = cosθ.

AMPLITUDE AND PERIOD OF THE SINE AND COSINE WAVES

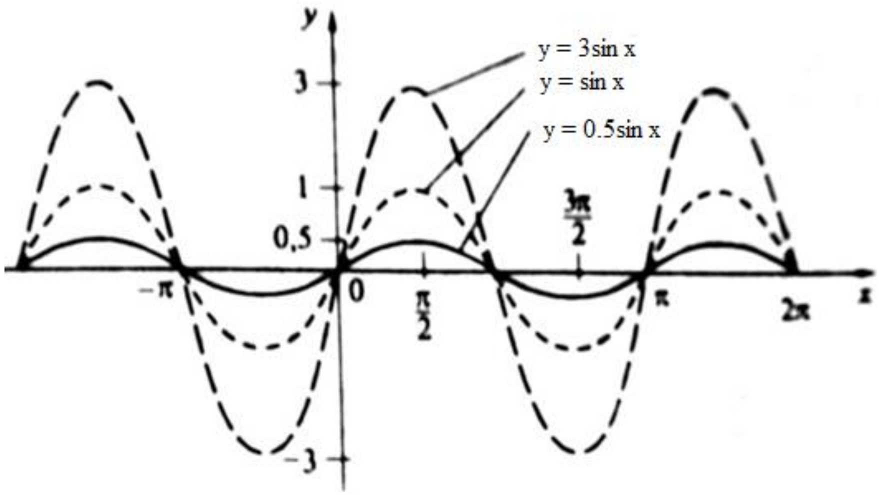




The amplitude of the sine wave refers to the vertical height of the wave. It is measured from the middle (midline) to the top of the wave. The period refers to the horizontal length of the wave.

You can manipulate the waves to make the altitude taller or shorter. You can manipulate the period to make the waves closer or farther apart. You can also shift the waves to the right or the left.

The figure below shows three sine waves. y = 3sin x, y = sin x, and y = 0.5sin x. Notice that the tallest one has a coefficient 3. This means that the sine wave is multiplied by 3, it is three times as tall. If there is a coefficient in front of the sine, that will indicate the amplitude. A number smaller than 1, like $\frac{1}{2}$ or any other positive fraction or decimal smaller than 1, will make the amplitude shorter. On the figure the graph y = 0.5sin x is half as tall as y = sin x. The larger the coefficient before the sine the taller the graph will be.



Sample Questions:

1. In the graph below A_2 = sin x. What could be the equation for A_1?

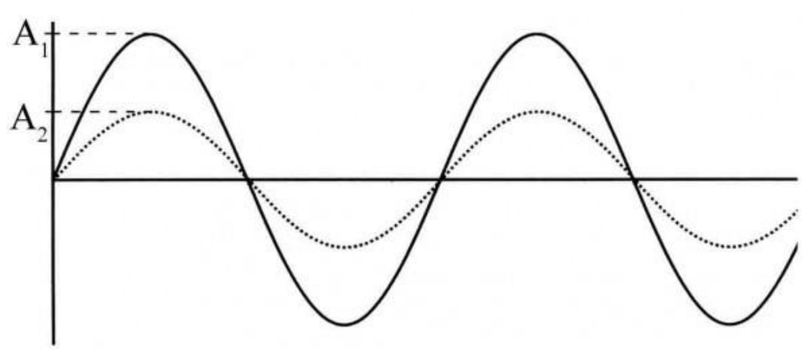
   A. A_1 = 2sin x

   B. A_1 = sin x

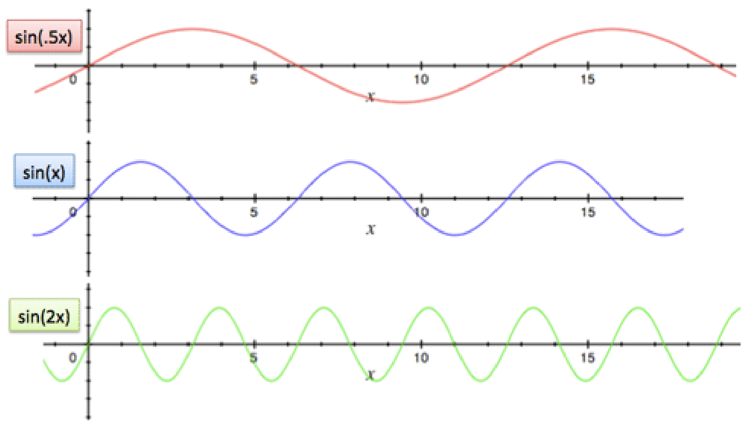
   C. A_1 = 0.5sin x

   D. A_1 = tan x

   E. cannot be determined



The following figure shows 3 sine waves with different wavelengths. The top figure is sin(0.5), the middle figure is sin(x), and the bottom figure is sin(2x). Notice that when the x value is multiplied by a larger number, the waves are faster, or closer together. When the x value is multiplied by a number smaller than 1, like $\frac{1}{2}$ or any other positive fraction or decimal smaller than 1, will make the waves slower or farther apart. Another name for the wavelength is the period.



The general equation for sine and cosine graphs are:

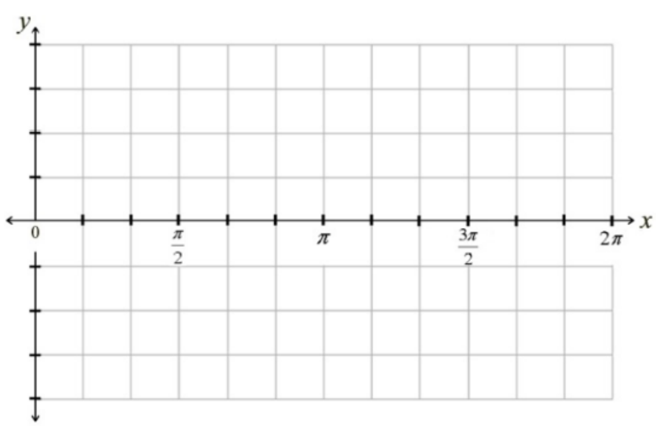
$f(x) = a \cdot \sin(bx) + k$ and $f(x) = a \cdot \cos(bx) + k$

- The **amplitude** $|a|$ is the distance from the midline to the top (or bottom) of the wave
- The **period** is calculated with the formula $\frac{2\pi}{|b|}$
- The **midline**, k, is the horizontal line that cuts the trigonometric function in half.

Sample Questions:

For the function $y = \sin x + 2$ find the following:

2. What is the amplitude?
3. What is the period?
4. What is the midline?
5. Sketch the graph



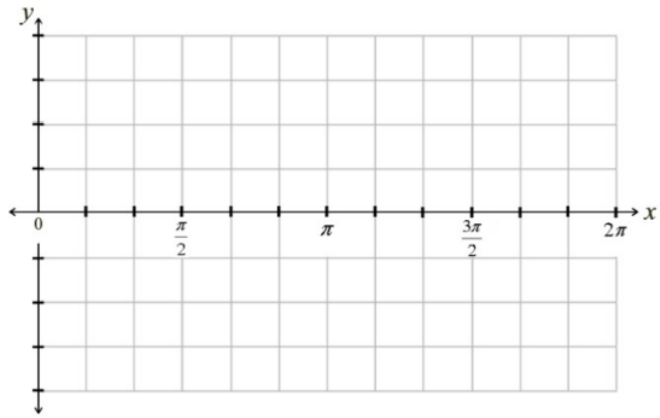
For the function $y = 3\sin(2x)$ find the following:

6. What is the amplitude?

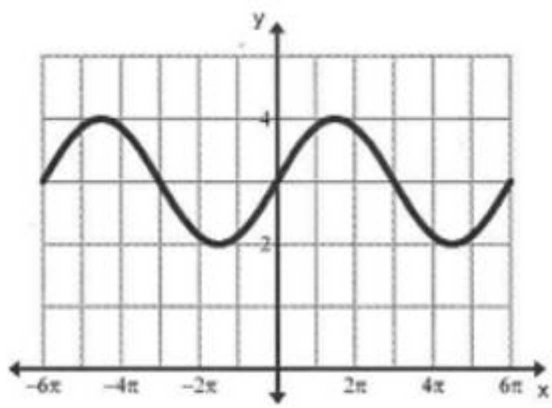
7. What is the period?

8. What is the midline?

9. Sketch the graph



For the graph at right, answer the following questions (the parent function is the sine graph):



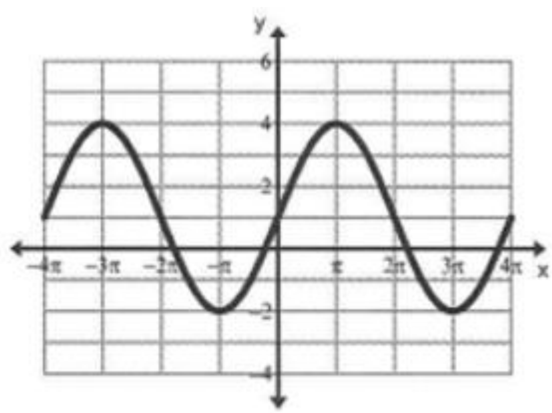
10. What is the midline?

11. What is the amplitude?

12. What is the period?

13. What is the equation of the graph?

For the graph at right, answer the following questions (the parent function is the sine graph):



14. What is the midline?

15. What is the amplitude?

16. What is the period?

17. What is the equation of the graph?

18. Compared to the graph of y = cos θ, the graph of y = 2 cos θ has: (Hint: this doesn't have a picture, but can you tell just from the equation? Did they modify the amplitude? Did they modify the wavelength or period?)

   A. twice the period and the same amplitude

   B. half the period and the same amplitude

   C. twice the period and half the amplitude

   D. twice the period and half the amplitude

   E. same period and twice and amplitude

19. The equations of the 2 graphs shown below are y_1(t) = $a_1\sin(b_1 t)$ and y_2(t) = $a_2\cos(b_2 t)$, where the constants b_1 and b_2 are both positive real numbers. Which of the following statements is true of the constants a_1 and a_2.

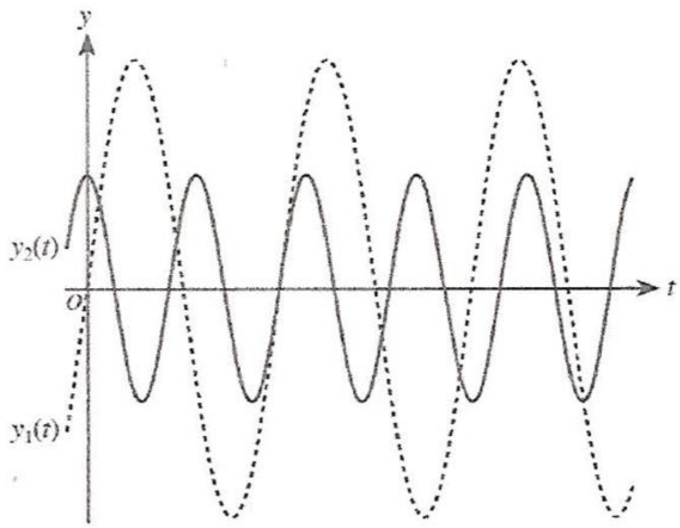
   A. $0 < a_1 < a_2$

   B. $0 < a_2 < a_1$

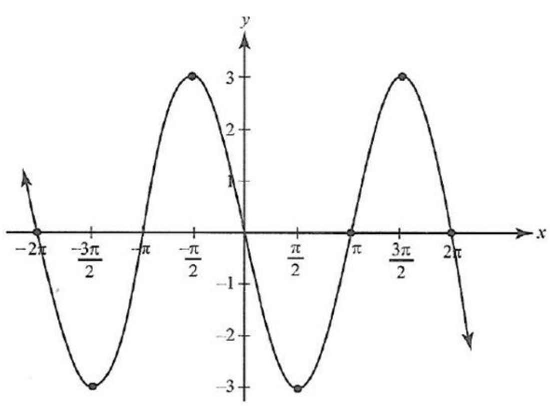
   C. $a_1 < 0 < a_2$

   D. $a_1 < a_2 < 0$

   E. $a_2 < a_1 < 0$



20. What is the amplitude of this function?



SIMPLIFYING TRIGONOMETRIC FUNCTIONS

Sometimes we have to simplify trigonometric functions. For example:

What is (cos x)(tan x)? In order to do this we re-write the function and look for anything that we can simplify or cancel out. Tan x can be re-written as $\frac{\sin x}{\cos x}$ which comes in really handy for simplifying.

(cos x)(tan x) start with the original question

$\left(\frac{\cos x}{1}\right)\left(\frac{\sin x}{\cos x}\right)$ re-write tan x as $\frac{\sin x}{\cos x}$

$\left(\frac{\cancel{\cos x}}{1}\right)\left(\frac{\sin x}{\cancel{\cos x}}\right)$ reduce fraction and simplify

$\sin x$ simplify

(cos x)(tan x) = sin x

Sample Questions:

21. Which trigonometric function (where defined) is equivalent to $\frac{sin^2 x}{(\cos x)(\tan x)}$?

A. $\frac{\cos x}{sin^2 x}$

B. $\frac{1}{\cos x}$

C. sin x

D. $\frac{1}{\sin x}$

E. $\frac{1}{sin^2 x}$

22. If g(x) = (csc x)(tan x), then which of the following trigonometric functions is equivalent to g(x)?
(Note: csc x = $\frac{1}{\sin x}$, sec x = $\frac{1}{\cos x}$, and cot x = $\frac{1}{\tan x}$)

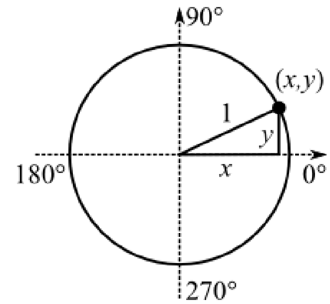
A. g(x) = sin x

B. g(x) = cos x

C. g(x) = tan x

D. g(x) = csc x

E. g(x) = sec x



When simplifying trigonometric functions there is another very handy tip to know. It's based on the unit circle and the Pythagorean theorem.

$a^2 + b^2 = c^2$ basic Pythagorean theorem
$x^2 + y^2 = 1$ plugging in data for this triangle
$\boldsymbol{sin^2x + cos^2x = 1}$ because sin = $\frac{opposite}{hypotenuse} = \frac{y}{1}$ = y and cos= $\frac{adjacent}{hypotenuse} = \frac{x}{1}$ = x.
{Note $\sin^2 x$ is another way of writing $(\sin x)^2$ which means $(\sin x)(\sin x)$, use whichever form works best for the situation. Ditto for $\cos^2 x$}

We can also rearrange this version of the Pythagorean theorem a few other ways
$sin^2x + cos^2x = 1$ begin with trig/Pythagorean theorem
$sin^2x = 1 - cos^2x$ subtract cos^2x from both sides
$\sqrt{(sinx)^2} = \sqrt{1 - cos^2x}$ take the square root of both sides
$\mathbf{sin}\, \boldsymbol{x} = \sqrt{\boldsymbol{1 - cos^2x}}$ you'd be surprised how often this comes in handy

$sin^2x + cos^2x = 1$ begin with trig/Pythagorean theorem
$cos^2x = 1 - sin^2x$ subtract sin^2x from both sides
$\sqrt{(cosx)^2} = \sqrt{1 - sin^2x}$ take the square root of both sides
$\mathbf{cos}\, \boldsymbol{x} = \sqrt{\boldsymbol{1 - sin^2x}}$ very useful information to simplify equations by substitution

For example: simplify the following trigonometric function: $(tanx)$ $(\sqrt{1 - sin^2x}\,)$. At first glance this looks terrifying, but by substituting it becomes a piece of cake.
$(tanx)$ $(\sqrt{1 - sin^2x}\,)$ start with the original function
$\left(\frac{sinx}{cosx}\right)\left(\frac{cosx}{1}\right)$ substitute $\left(\frac{sinx}{cosx}\right)$ = tan and $\cos x = \sqrt{1 - sin^2x}$
$\left(\frac{sinx}{\cancel{cosx}}\right)\left(\frac{\cancel{cosx}}{1}\right)$ reduce fraction and simplify
$sinx$ simplify
$(tanx)$ $(\sqrt{1 - sin^2x}\,)$ = $sinx$ that looks so much better

Sample Questions:

23. If x = sinθ, y = cosθ, and z = tanθ, then $x^2 + y^2$ = ?

A. z^2
B. $(x + y)^2$
C. 2y
D. 0
E. 1

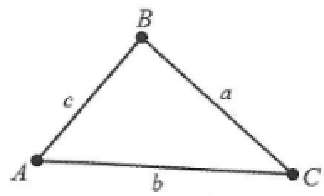
24. For x such that 0 < x < $\frac{\pi}{2}$, the expression $\frac{\sqrt{1 - cos^2x}}{sinx} + \frac{\sqrt{1 - sin^2x}}{cosx}$ is equivalent to:

A. 0
B. 1
C. 2
D. -tan x
E. sin2x

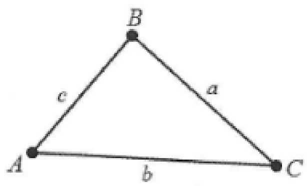
THE LAW OF SINES

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

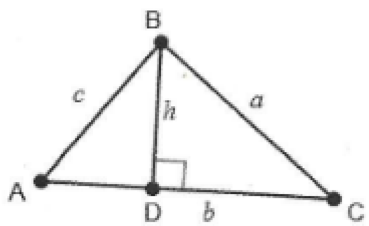


For any ΔABC, the Law of Sines relates the sine of each angle to the length of the side opposite the angle.



Proof:

Start with any triangle that has angles A, B, and C and side lengths, a, b, and c, where 'a' is the side oposite angle A, 'b' is the side opposite angle B, and 'c' is the side oposite angle C.



Construct an altitude, h, from one of the angles to the side opposite the angle. The two triangles formed, ΔABD and ΔBCD, are right triangles.

$\sin A = \frac{h}{c}$ and $\sin C = \frac{h}{a}$ Use the definition of sine to relate the base angles, angle A and angleC, to the hypotenuse of each and the altitude.

$c\sin A = h$ and $a\sin C = h$ Use the multiplication property of equality to solve each equation for h.

$c\sin A = a\sin C$ Use the transitive property of equality to set the equations equal to one another.

$\frac{\sin A}{a} = \frac{\sin C}{c}$ Use the division property of equality

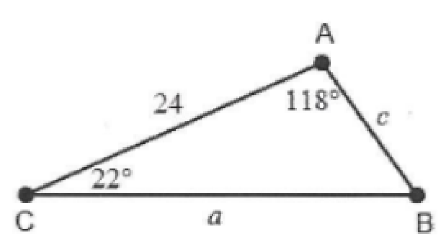
Sample Questions:

You can use the Law of Sines to solve unknown information about a triangle.

For the triangle ΔABC at right:

25. What is the measure of angle B?

Using the law of sines $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$



26. What is the length of side a?

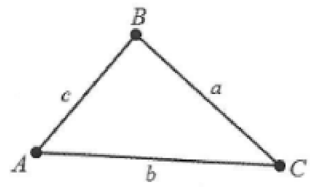
27. What is the length of side c?

LAW OF COSINES

$\boldsymbol{a^2 = b^2 + c^2 - 2bc}\textbf{cosA}$

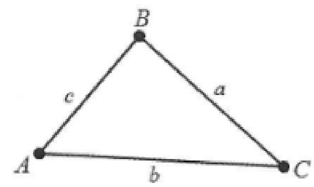
$\boldsymbol{b^2 = a^2 + c^2 - 2ac}\textbf{cosB}$

$\boldsymbol{c^2 = a^2 + b^2 - 2ab}\textbf{cosC}$

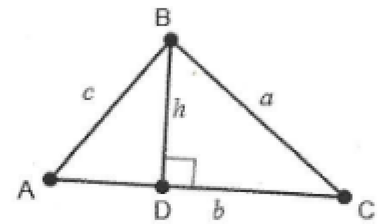


For any triangle ABC, the Law of Cosines relates the length of a side to the other two sides of a triangle and the cosine of the included angle.

Proof:
Start with any triangle that has angles A, B, and C and side lengths, a, b, and c, where 'a' is the side oposite angle A, 'b' is the side opposite angle B, and 'c' is the side oposite angle C.



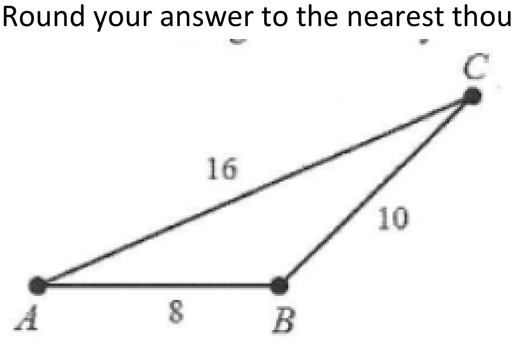
Construct an altitude, h, from one of the angles to the side opposite the angle. The two triangles formed, ΔABD and ΔBCD, are right triangles.



$a^2 = (b - x)^2 + h^2$ Use the Pythagorean theorem to relate the sides of triangle ADB
$a^2 = b^2 - 2bx + x^2 + h^2$ Expand the binomial
$a^2 = b^2 - 2b[c(cosA)] + c^2$ Use cosA = $\frac{x}{c}$ or c(cosA) = x and $x^2 + h^2 = c^2$
$a^2 = b^2 + c^2 - 2bc\cos A$ Rearrange the terms
The Law of Cosines, like the Law of Sines, can be used to solve unknown information about a triangle.

Sample Questions:

28. Use the Law of Cosines $\boldsymbol{b^2 = a^2 + c^2 - 2ac}\textbf{cosB}$ to find the measure of angle B in the triangle below. Round your answer to the nearest thousandth.



29. Using either Law of Cosines or Law of Sines find the measure of angle A in the triangle above.

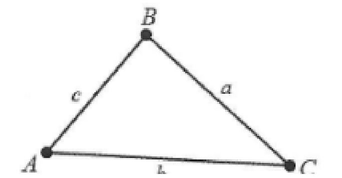
30. Using any method you choose, find the measure of angle C in the triangle above.

AREA OF A TRIANGLE **Area =** $\frac{1}{2}bc(sinA)$

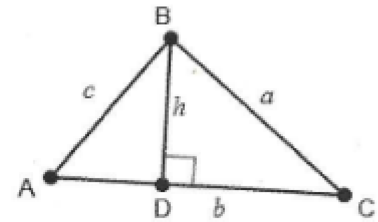
We already know that the area of a triangle is $\frac{1}{2}(base)(height)$, but what if the height is unknown? Using trigonometry, we can find the height of any triangle, and can therefore find the area of any triangle.

Proof:

Start with any triangle that has angles A, B, and C and side lengths, a, b, and c, where 'a' is the side oposite angle A, 'b' is the side opposite angle B, and 'c' is the side oposite angle C.



Construct an altitude, h, from one of the angles to the side opposite the angle. The two triangles formed, ΔABD and ΔBCD, are right triangles.



$\sin A = \frac{h}{c}$ Find the measure of h in terms of angle A and side c using the sine ratio for angle A

$c\sin A = h$ Multiply both sides by c

Area = $\frac{1}{2}(base)(height)$ The base of the entire triangle is b and the height is h.

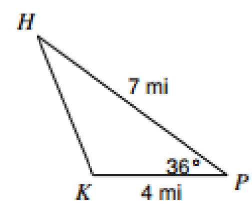
Area = $\frac{1}{2}(b)(csinA)$ Substitute csinA = h into the area formula for h, the height

Area = $\frac{1}{2}bc(sinA)$ Rearrange the formula.

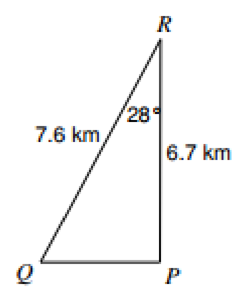
This formula works for acute, obtuse, and right angles.

Sample Questions:

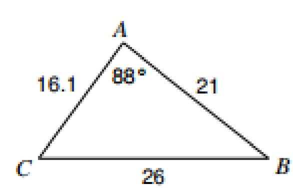
31. Find the area of triangle HKP.



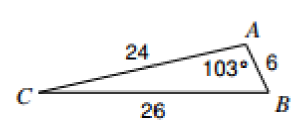
32. Find the area of triangle QRP.



33. Find the area of triangle ABC.



34. Find the area of triangle ABC.



Answers

1. A
2. 1
3. 2π
4. 2
5. see graph at right
6. 3
7. π
8. 0
9. see graph at right
10. 3
11. 1
12. 6 π
13. y = $\sin\left(\frac{1}{3}x\right)$ + 3
14. 1
15. 3
16. 4 π
17. y = $3\sin\left(\frac{1}{2}x\right)$ + 1
18. E
19. B
20. 3
21. C
22. E
23. E
24. C
25. 40°
26. 32.967
27. 13.987
28. 125.1°
29. 30.753°
30. 24.147°
31. 8.2 mi^2
32. 12km^2
33. 168.947$units^2$
34. 70.155$units^2$