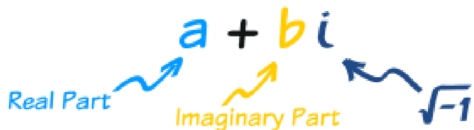
Name:______________________________________ Date:___________________________

Math 3 Variable Manipulation Part 4
Polynomials B

COMPLEX NUMBERS

A Complex Number is a combination of a Real Number and an Imaginary Number:



Examples:

1 + i	39 + 3i	0.8 − 2.2i	−2 + πi	√2 + i/2

A Complex Number is just two numbers added together (a Real and an Imaginary Number). But either part can be 0, so all Real Numbers and Imaginary Numbers are also Complex Numbers.

Complex Number	Real Part	Imaginary Part
3 + 2i	3	2
5	5	0
−6i	0	−6

Add & Subtract Complex Numbers

To add two complex numbers, add each part separately (**remember:** numbers must be the same type to add or subtract).

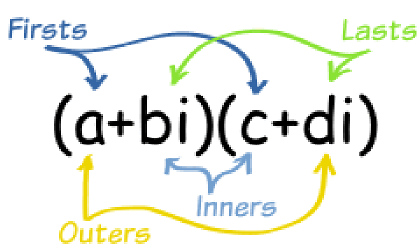
(a+bi) + (c+di) = (a+c) + (b+d)i

Example: (3 + 2i) + (1 + 7i) = (4 + 9i)

Multiplying Complex Numbers

Each part of the first complex number gets multiplied by each part of the second complex number

Just use "FOIL", which stands for "Firsts, Outers, Inners, Lasts"



Firsts: a × c
Outers: a × di
Inners: bi × c
Lasts: bi × di

$$(a+bi)(c+di) = ac + adi + bci + bdi^2$$

Example: (3 + 2i)(1 + 7i)

Solution:

$$\begin{aligned}(3 + 2i)(1 + 7i) &= 3\times1 + 3\times7i + 2i\times1 + 2i\times7i \\ &= 3 + 21i + 2i + 14i^2 \\ &= 3 + 21i + 2i - 14 \qquad (\text{because } i^2 = -1) \\ &= -11 + 23i\end{aligned}$$

Example: $(1 + i)^2$

Solution:

$$
\begin{aligned}
(1 + i)2 = (1 + i)(1 + i) &= 1\times1 + 1\times i + 1\times i + i^2 \\
&= 1 + 2i - 1 \qquad \text{(because } i^2 = -1) \\
&= 0 + 2i
\end{aligned}
$$

But There is a Quicker Way!

Use this rule:

$(a+bi)(c+di) = (ac-bd) + (ad+bc)i$

Example: $(3 + 2i)(1 + 7i)$

Solution: $(3 + 2i)(1 + 7i) = (3\times1 - 2\times7) + (3\times7 + 2\times1)i = -11 + 23i$

Why Does That Rule Work?

It is just the "FOIL" method after a little work:

$$
\begin{aligned}
(a+bi)(c+di) &= ac + adi + bci + bdi^2 && \text{FOIL method} \\
&= ac + adi + bci - bd && \text{(because } i^2 = -1) \\
&= (ac - bd) + (ad + bc)i && \text{(gathering like terms)}
\end{aligned}
$$

And there we have the $(ac - bd) + (ad + bc)i$ pattern.

This rule is certainly faster, but if you forget it, just remember the FOIL method.

Another attribute of complex numbers is that $i^2 = -1$. This can be proven through multiplication.

Example: i^2

Solution: i can also be written with a real and imaginary part as 0 + i

$$
\begin{aligned}
i^2 = (0 + i)2 &= (0 + i)(0 + i) \\
&= (0\times0 - 1\times1) + (0\times1 + 1\times0)i \\
&= -1 + 0i \\
&= -1
\end{aligned}
$$

And that agrees nicely with the definition that $i^2 = -1$

Dividing Complex Numbers

To take divide complex numbers, multiply both numerator and denominator by the complex conjugate.

Example: What is $\frac{2+5i}{3+i}$ as a single complex number?

Solution: Multiply both numerator and denominator by the complex conjugate of 3 + i (which is 3 - i)

$$\frac{2+5i}{3+i} = \frac{2+5i}{3+i} \times \frac{3-i}{3-i} = \frac{(6+5)+(-2+15)i}{9+1} = \frac{11+13i}{10} = \frac{11}{10} + \frac{13}{10}i$$

Your calculator can be a really handy tool for problems dealing with imaginary and irrational numbers.

Sample Questions:

1. $4i(-2 - 8i)$

2. $(1 - 7i)^2$

3. $6(-7 + 6i)(-4 + 2i)$

4. $6i \cdot -4i + 8$

5. $(2 + i)(2 - i)$?

6. $(3 + i)^2$?

7. What is $(3 - 5i) + (-4 + 7i)$?

8. What is $\frac{7-3i}{5-2i}$ as a single complex number?

9. In the complex numbers, where $i^2 = -1$, $\frac{1}{(1+i)} \times \frac{(1-i)}{(1+i)} = $?

10. For $i^2 = -1$, $(4 + i)^2 = $?

11. What is the product of the complex numbers $(-3i + 4)$ and $(3i + 4)$?

12. For the imaginary number i, which of the following is a possible value of i^n if n is an integer less than 5?

A. 0
B. -1
C. -2
D. -3
E. -4

SOLVING A QUADRATIC EQUATION

To solve a quadratic equation, put it in the form of $ax^2 + bx + c = 0$ -- in other words, set it equal to 0. Then **factor** the left side (if you can), and set each factor equal to 0 separately to get the two solutions.

Sometimes the left side might not be obviously factorable. You can always use the quadratic formula. Just plug in the coefficients a, b, and c from $ax^2 + bx + c = 0$ into the formula:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example: Solve $x^2 + 4x + 2 = 0$

Solution: To solve $x^2 + 4x + 2 = 0$, plug a = 1, b = 4, and c = 2 into the formula:

$$\frac{-4 \pm \sqrt{4^2 - 4 * 1 * 2}}{2 * 1}$$

$$\frac{-4 \pm \sqrt{8}}{2} = -2 \pm \sqrt{2}$$

Whether you use reverse-FOIL or the quadratic equation, you will almost always get two solutions, or roots, to the equation.

Sample Questions:

13. $2x^2 + 3x - 20 = 0$

14. $4b^2 + 8b + 7 = 4$

15. $5x^2 + 9x = -4$

16. $9n^2 = 4 + 7n$

17. $2x^2 - 36 = x$

18. $k^2 - 31 - 2k = -6 - 3k^2 - 2k$

COMPLEX NUMBERS IN QUADRATIC EQUATIONS

Example: What are the roots of the quadratic equation $x^2 - 2x + 3 = 0$

Solution: Use the quadratic equation formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with a = 1, b = -2 and c = 3

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 1 \times 3}}{2 \times 1} = \frac{2 \pm \sqrt{-8}}{2} = 1 + i\sqrt{2} \text{ or } 1 - i\sqrt{2}$$

Sample Questions:

19. What are the roots of the quadratic equation $x^2 + 4x + 7 = 0$?

20. Solve: $8a^2 + 6a = -5$

21. Solve: $10x^2 + 9 = x$

POLYNOMIAL IDENTITIES

Polynomial Identities	
Perfect Square Trinomial $(A+B)^2 = A^2 + 2AB + B^2$	$(4x+3y)^2 = (4x)^2 + 2(4x)(3y) + (3y)^2$ $= 16x^2 + 24xy + 9y^2$
Difference of Squares $(A+B)(A-B) = A^2 - B^2$	$(2x+5y)(2x-5y) = (2x)^2 - (5y)^2$ $= 4x^2 - 25y^2$
Cubic Polynomials $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$ $(A-B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$	$(2x+5y)^3 = (2x)^3 + (3)(2x)^2(5y) + (3)(2x)(5y)^2 + (5y)^3$ $= 8x^3 + 60x^2y + 150xy^2 + 125y^3$ $(2x-5y)^3 = (2x)^3 - (3)(2x)^2(5y) + (3)(2x)(5y)^2 - (5y)^3$ $= 8x^3 - 60x^2y + 150xy^2 - 125y^3$
Sum and Difference of Cubes $A^3 + B^3 = (A+B)(A^2 - AB + B^2)$ $A^3 - B^3 = (A-B)(A^2 + AB + B^2)$	$27x^3 + 64y^3 = (3x+4y)\left[(3x)^2 - (3x)(4y) + (4y)^2\right]$ $= (3x+4y)(9x^2 - 12xy + 16y^2)$ $27x^3 - 64y^3 = (3x-4y)\left[(3x)^2 + (3x)(4y) + (4y)^2\right]$ $= (3x-4y)(9x^2 + 12xy + 16y^2)$
Trinomial Leading Coefficient 1 $x^2 + (a+b)x + ab = (x+a)(x+b)$	$x^2 + 5x + 6 = x^2 + (2+3)x + (2)(3)$ $= (x+2)(x+3)$ $x^2 - 5x + 6 = x^2 + (-2-3)x + (-2)(-3)$ $= (x-2)(x-3)$

Quadratic Formula Given $ax^2 + bx + c = 0$ $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	$2x^2 - 4x - 5 = 0$ $x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(-5)}}{2(2)}$ $= \dfrac{4 \pm \sqrt{16+40}}{4}$ $= \dfrac{4 \pm \sqrt{56}}{4}$ $= \dfrac{4 \pm 2\sqrt{14}}{4}$ $= \dfrac{2 \pm \sqrt{14}}{2}$	$4x^2 + 9$ $x = \dfrac{-0 \pm \sqrt{0^2 - 4(4)(9)}}{2(4)}$ $= \dfrac{\pm\sqrt{-144}}{8}$ $= \dfrac{\pm 12i}{8}$ $= \dfrac{\pm 3i}{2}$
Sum of Squares $A^2 + B^2 = (A+Bi)(A-Bi)$	$4x^2 + 9 = (2x+3i)(2x-3i)$	

A polynomial with integer coefficients that cannot be factored into polynomials of lower degree, also with integer coefficients, is called an irreducible or prime polynomial.

Sample Questions:
Solve the following

22. $(4x - y)^3$

23. $(10x + 4i)(10x - 4i)$

24. $(x - 11y)(x^2 + 11xy + 12y^2)$

Factor the expressions using the polynomial identities.

25. $27x^3 - y^3$

26. $27x^3 - 54x^2y + 36xy^2 - 8y^3$

27. $343x^3 + 8y^3$

28. $144x^2 - 25$

29. $x^2 + 4x - 45$

Binomial Theorem

To understand binomial theorem, first note the following example problems

Example 1. Expand $(x+1)^0 = 1$

Example 2. Expand $(x+1)^1 = x + 1$

Example 3. Expand $(x+1)^2 = x^2 + 2x + 1$

Example 4. Expand $(x+1)^3 = x^3 + 3x^2 + 3x + 1$

Example 5. Expand $(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$

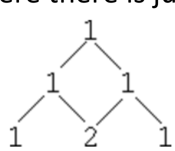
Example 6. Write the coefficients of each expansion.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

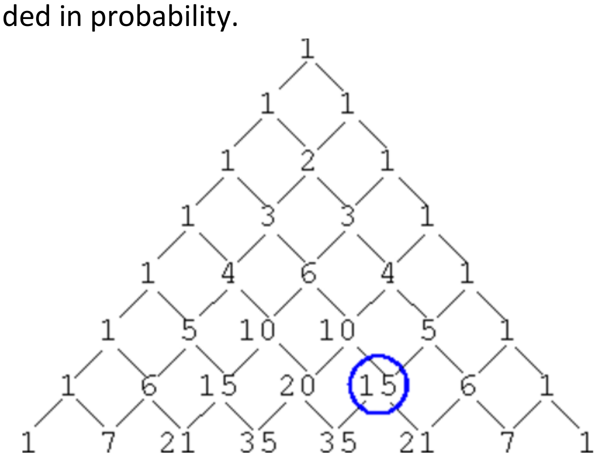
These numbers form what is called Pascal's Triangle. It was named in honor of Blaise Pascal. The pattern in the triangle represents the coefficients for a binomial expansion.

Row 0 1
Row 1 1 1
Row 2 1 2 1
Row 3 1 3 3 1
Row 4 1 4 6 4 1
Row 5 1 5 10 10 5 1

To easily make this pattern, add two consecutive numbers and put the sum underneath and between the two numbers. (Note: Where there is just one number, carry down the 1 to the next line.)



These numbers can be represented by the value of "${}_nC_r$". For example, to find ${}_6C_4$, you go down to the row where there is a "6" after the initial "1", ignore that initial "1" and then go over to the 4th entry, to find that ${}_6C_4 = 15$. Remember to ignore the initial "1" when counting over! If you ignore the top "1" in the triangle, it is also the 6th row to get the 6. Note: you can also calculate "${}_nC_r$" using sums and factorials which is included in probability.



An exponent of 2 means to multiply by itself:
(a+b)2 = (a+b)(a+b) = a2 + 2ab + b2

For an exponent of 3 just multiply again:
(a+b)3 = (a2 + 2ab + b2)(a+b) = a3 + 3a2b + 3ab2 + b3

The Pattern: The exponents of a start at 3 and go down: 3, 2, 1, 0:

$$a^3 + 3a^2b + 3a\,b^2 + b^3$$

3 2 1 0

Likewise the exponents of b go upwards: 0, 1, 2, 3:

$$a^3 + 3a^2b + 3a\,b^2 + b^3$$

0 1 2 3

The coefficients come from Pascal's triangle or "n Choose k".

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

It is commonly called "n choose k" because it is how many ways to choose k elements from a set of n.
The "!" means "factorial", for example 4! = 4×3×2×1 = 24
Figure out each coefficient by starting with n = highest power and k = 0 and move iteratively k = 1, k = 2, until k = n. That will make n + 1 coefficients which is how many terms there are.

The Binomial Theorem states that for any positive integer n,
$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n}b^n$, where $\binom{n}{r} = {}_nC_r$. Although the binomial theorem uses combinations to determine the coefficients for each term, Pascal's triangle can be used to determine the coefficients instead.

Notice the a exponents decrease from n to 0, while the b exponents increase from 0 to n.

According to Binomial Theorem:

$(a + b)^0 = 1$
$(a + b)^1 = a^1b^0 + a^0b^1$
$(a + b)^2 = a^2b^0 + 2a^1b^1 + a^0b^2$
$(a + b)^3 = a^3b^0 + 3a^2b^1 + 3a^1b^2 + a^0b^3$
$(a + b)^4 = a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + a^0b^4$
$(a + b)^5 = a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + a^0b^5$

Example: Expand $(3x – 2y)^5$
Solution:

$(3x-2y)^5$: *Use* $(a+b)^5 = a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + a^0b^5$ *where* $a = 3x$ and $b = -2y$

$$(3x)^5(-2y)^0 + 5(3x)^4(-2y)^1 + 10(3x)^3(-2y)^2 + 10(3x)^2(-2y)^3 + 5(3x)^1(-2y)^4 + (3x)^0(-2y)^5$$
$$243x^5 + 5(81x^4)(-2y) + 10(27x^3)(4y^2) + 10(9x^2)(-8y^3) + 5(3x)(16y^4) + (-32y^5)$$
$$243x^5 - 810x^4y + 1080x^3y^2 - 720x^2y^3 + 240xy^4 - 32y^5$$

Sample Questions:

Expand each of the binomials using the Binomial Theorem.

30. $(x + 3)^6$

31. $(2x - 1)^5$

32. $(x^4 - y)^5$

Answer Key

1. $32 - 8i$
2. $-48 - 14i$
3. $96 - 228i$
4. 32
5. 5
6. $8 + 6i$
7. $-1 + 2i$
8. $\frac{41-i}{29}$ or $\frac{41}{29} - \frac{1}{29}i$
9. $\frac{(i+1)}{-2}$ or $\frac{(-i-1)}{2}$ or $\frac{(-1-i)}{2}$
10. $15 + 8i$
11. 25
12. -1 (B)
13. $\{\frac{5}{2}, -4\}$
14. $\{-\frac{1}{2}, -\frac{3}{2}\}$
15. $\{-\frac{4}{5}, -1\}$
16. $\{\frac{7+\sqrt{193}}{18}, \frac{7-\sqrt{193}}{18}\}$
17. {9/2, -4}
18. {-5/2, 5/2}
19. $-2 + i\sqrt{3}$ or $-2 - i\sqrt{3}$

20. $\{\frac{-3+i\sqrt{31}}{8}, \frac{-3-i\sqrt{31}}{8}\}$

21. $\{\frac{1+i\sqrt{359}}{20}, \frac{1-i\sqrt{359}}{20}\}$

22. $64x^3 - 48x^2y + 12xy^2 - y^3$
23. $100x^2 + 16$
24. $x^3 + 12x - 121xy^2 - 132y$
25. $(3x - y)(9x^2 + 3xy + y^2)$
26. $(3x - 2y)^3$
27. $(7x + 2y)(49x^2 - 14xy + 4y^2)$
28. $(12x - 5i)(12x + 5i)$
29. $(x + 9)(x - 5)$
30. $(x + 3)^6 = x^6 + 18x^5 + 135x^4 + 540x^3 + 1215x^2 + 1458x + 729$
31. $(2x - 1)^5 = 32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$
32. $(x^4 - y)^5 = x^{20} - 5x^{16}y + 10x^{12}y^2 - 10x^8y^3 + 5x^4y^4 - y^5$