Name:____________________________________ Date:__________________________

Math 2 Variable Manipulation Part 2
Powers & Roots

PROPERTIES OF EXPONENTS:

Properties of Exponents (All bases are non-zero)	**Properties of Rational Exponents** (All bases are non-zero)	**Examples**
$x^a \cdot x^b = x^{a+b}$	$x^{\frac{p}{q}} \cdot x^{\frac{r}{s}} = x^{\frac{p}{q}+\frac{r}{s}}$	$x^{\frac{3}{4}} \cdot x^{\frac{1}{5}} = x^{\frac{3}{4}+\frac{1}{5}} = x^{\frac{19}{20}}$
$\dfrac{x^a}{x^b} = x^{a-b}$	$\dfrac{x^{\frac{p}{q}}}{x^{\frac{r}{s}}} = x^{\frac{p}{q}-\frac{r}{s}}$	$\dfrac{x^{\frac{2}{3}}}{x^{\frac{3}{5}}} = x^{\frac{2}{3}-\frac{3}{5}} = x^{\frac{1}{15}}$
$x^0 = 1$		$\dfrac{x^a}{x^a} = x^{a-a} = x^0$ However: $\dfrac{x^a}{x^a} = 1$ Therefore: $x^0 = 1$
$x^{-a} = \dfrac{1}{x^a}$	$x^{-\frac{p}{q}} = \dfrac{1}{x^{\frac{p}{q}}}$	$x^{-\frac{2}{5}} = \dfrac{1}{x^{\frac{2}{5}}}$
$(xy)^n = x^n y^n$	$(xy)^{\frac{p}{q}} = x^{\frac{p}{q}} y^{\frac{p}{q}}$	$(xy)^{\frac{3}{4}} = x^{\frac{3}{4}} y^{\frac{3}{4}}$
$(x^m)^n = x^{m \cdot n}$	$\left(x^{\frac{p}{q}}\right)^{\frac{r}{s}} = x^{\frac{p}{q} \cdot \frac{r}{s}}$	$\left(x^{\frac{3}{4}}\right)^{\frac{2}{3}} = x^{\frac{3}{4} \cdot \frac{2}{3}} = x^{\frac{2}{4}} = x^{\frac{1}{2}}$
$\left(\dfrac{x}{y}\right)^m = \dfrac{x^m}{y^m}$	$\left(\dfrac{x}{y}\right)^{\frac{p}{q}} = \dfrac{x^{\frac{p}{q}}}{y^{\frac{p}{q}}}$	$\left(\dfrac{x}{y}\right)^{\frac{2}{3}} = \dfrac{x^{\frac{2}{3}}}{y^{\frac{2}{3}}}$
$\left(\dfrac{x^m}{y^n}\right)^{-1} = \dfrac{x^{-m}}{y^{-n}} = \dfrac{y^n}{x^m}$	$\left(\dfrac{x^{\frac{m}{n}}}{y^{\frac{a}{b}}}\right)^{-1} = \dfrac{x^{-\frac{m}{n}}}{y^{-\frac{a}{b}}} = \dfrac{y^{\frac{a}{b}}}{x^{\frac{m}{n}}}$	$\left(\dfrac{x^{\frac{1}{2}}}{y^{\frac{3}{4}}}\right)^{-1} = \dfrac{x^{-\frac{1}{2}}}{y^{-\frac{3}{4}}} = \dfrac{y^{\frac{3}{4}}}{x^{\frac{1}{2}}}$

EXPONENT REVIEW PROBLEMS:

1. $2x^3 + 3x - x^3 + 4x + 5 = ?$

2. $(3x^2 + 4x) - (3x + 2) = ?$

3. The expression $-8x^3(7x^6 - 3x^5)$ is equivalent to ?

4. $(2x^4y)(3x^5y^8)$ is equivalent to ?

5. For all x > 0, the expression $\frac{3x^3}{3x^9}$ equals:

6. Reduce $\frac{x^8y^{12}}{x^4y^3z^2}$ to its simplest terms.

7. $(3x^3)^3$ is equivalent to ?

8. If $xyz \neq 0$, which of the following is equivalent to
$$\frac{x^2y^3z^4}{(xyz^2)^2}\ ?$$
   a. 1/y
   b. 1/z
   c. y
   d. x/yz
   e. xyz

9. What is the value of the expression $2x^3 - x^2 + 3x + 5$ for x = -2?

MATH 2 LEVEL:

ALGEBRA PROBLEMS WITH EXPONENTS

Use algebra and exponent rules to solve math 2 level problems.

Example: What is the value of $(5^{1/3})^3$?

Solution: $(5^{1/3})^3 = 5^{(1/3)3} = 5^{3/3} = 5$

Example: What is the value of $(a^8)^{3/2}$?

Solution: $(a^8)^{3/2} = a^{(8*3/2)} = a^{24/2} = a^{12}$

Sample Questions:

10. $(9r^4)^{0.5}$

11. $(81x^{12})^{1.25}$

12. $(216r^9)^{1/3}$

13. $\frac{2x^{-7/4}}{4x^{4/3}}$

14. $\frac{3x^{-1/2}\, 3x^{1/2}\, y^{-1/3}}{3y^{-7/4}}$

15. $\left(\frac{x^{1/2}\, y^{-2}}{yx^{-7/4}} \right)^4$

16. If $\frac{n^x}{n^y} = n^2$ for all $n \neq 0$, which of the following must be true?
    a. $x + y = 2$
    b. $x - y = 2$
    c. $x \times = 2$
    d. $x \div y = 2$
    e. $\sqrt{xy} = 2$

17. If $n^x * n^8 = n^{24}$ and $(n^6)^y = n^{18}$, what is the value of $x + y$?

MATH 1 REVIEW:

ADDING AND SUBTRACTING ROOTS

You can add or subtract radical expressions only if the part under the radicals is the same. In other words, treat it like a variable. Just like 2x + 3x would equal 5x:

$2\sqrt{3} + 3\sqrt{3} = 5\sqrt{3}$

In other words, you can only add or subtract the numbers in front of the square root sign the numbers under the sign stay the same.

MULTIPLYING AND DIVIDING ROOTS

You can distribute the square root sign over multiplication and division. The product of square roots is equal to the square root of the product:

$\sqrt{3}$ x $\sqrt{5}$ = $\sqrt{3x5}$ = $\sqrt{15}$

The quotient of square roots is equal to the square root of the quotient:

$\frac{\sqrt{6}}{\sqrt{3}} = \sqrt{\frac{6}{3}} = \sqrt{2}$

SIMPLIFYING SQUARE ROOTS

To simplify a square root**, factor out the perfect squares** under the radical, square root them, and put the result in front of the part left under the square root sign (the non-perfect- square factors):

$\sqrt{12} = \sqrt{4 \times 3} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3}$

To solve for a power of two, take the square root. To solve for a power of three, take the cube root, etc.:

x^2 = 16. Take the square root of both sides: $\sqrt{x^2}$ = $\sqrt{16}$ becomes x = 4

Sample Problems:

18. $3\sqrt{5}$ + 6 $\sqrt{5}$ = ?

19. $\frac{8\sqrt{18}}{4\sqrt{2}}$ = ?

20. $3\sqrt{3}$ (4 - $3\sqrt{5}$) = ?

21. $\sqrt{18}$ = ?

MATH 2 LEVEL:
PROPERTIES OF ROOTS/RADICALS:

A square is a number times itself: (5)(5) = 5^2. To solve for squares, use a square root. $\sqrt{25} = 5$. A cube is a number multiplied by itself three times: (5)(5)(5) = 5^3. To solve for cubes, use a cube root. $\sqrt[3]{125} = 5$.

Properties of Radicals
$\sqrt[n]{a^n} = a$ $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Simplifying Radicals:
Radicals that are simplified have:

- no fractions left under the radical.
- no perfect power factors in the radicand, *k*.
- no exponents in the radicand, *k*, greater than the index, *n*.

Example: Simplify $\sqrt[3]{1000}$
Solution: 10*10*10 = 1000 so $\sqrt[3]{1000}$ = 10

Example: Simplify $\sqrt[3]{24m^3}$
Solution: Find all the multiples of 3 under the radical and pull out the cubes.
$\sqrt[3]{24m^3}$ = $\sqrt[3]{24}$ * $\sqrt[3]{m^3}$ = $\sqrt[3]{2*2*2*3}$ * $\sqrt[3]{m^3}$ = $\sqrt[3]{2^3}$ * $\sqrt[3]{3}$ * $\sqrt[3]{m^3}$ = 2m$\sqrt[3]{3}$

Example: Simplify $\sqrt[3]{56x^3y}$
Solution: As in the previous example, find all the multiples of 3 under the radical and pull out the cubes.
$\sqrt[3]{56x^3y}$ = $\sqrt[3]{2*2*2*7}$* $\sqrt[3]{x^3}$ * $\sqrt[3]{y}$ = 2$\sqrt[3]{7}$ * x * $\sqrt[3]{y}$ = 2x$\sqrt[3]{7y}$

Sample Questions:

22. $\sqrt[3]{162}$

23. Simplify $\sqrt[3]{16x^3y^8}$

24. Simplify $\sqrt[4]{405x^3y^5}$

25. Simplify $\sqrt[5]{224x^7}$

MORE PROPERTIES OF ROOTS

A radical can be written as a term with a rational exponent. For example, $x^{\frac{1}{n}} = \sqrt[n]{x}$ where n is an integer and n ≠ 0 . In general, $x^{\frac{m}{n}} = \sqrt[n]{x^m}$ where m and n are integers and n ≠ 0. $x^{\frac{m}{n}}$ can also be written as $(\sqrt[n]{x})^m$.

$$x^{\frac{m}{n}} = \sqrt[n]{x^m}$$

Example: Write the expression in exponential form. $(\sqrt[3]{3m})^4$
Solution: Since it is a cube root, the exponent is of the form 1/3 and since the entire cube it taken to the 4th power, the exponent is 4/3. So the answer is $(3m)^{4/3}$.

Example: Write the expression in radical form. $(5a)^{-5/4}$
Solution: Since it is of the form ¼, it is a 4th root, and since it is a 5, then the entire 4th root is taken to the 5th power, and since the exponent is negative, the entire root part is in the denominator. So the answer is $\frac{1}{\sqrt[4]{5a^5}}$.

Sample Questions:

26. Express in radical form: $(10y)^{3/2}$

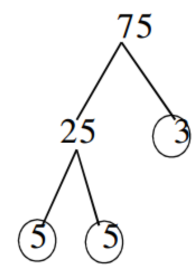
27. Express in exponential form: $\sqrt{6r}$

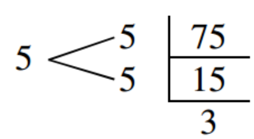
28. Express in radical form: $(5m)^{-1/2}$

29. Express in exponential form: $(\sqrt[4]{2})^5$

SIMPLIFY RADICALS

<table>
<tr><td>Method 1:
Find Perfect Squares Under the Radical
1. Rewrite the radicand as factors that are perfect squares and factors that are not perfect squares.
2. Rewrite the radical as two separate radicals.
3. Simplify the perfect square.</td><td>Example: $\sqrt{75}$
$\sqrt{25 \cdot 3}$
$\sqrt{25} \cdot \sqrt{3}$
$5\sqrt{3}$</td></tr>
</table>

<table>
<tr><td>Method 2:
Use a Factor Tree
1. Work with only the radicand.
2. Split the radicand into two factors.
3. Split those numbers into two factors until the number is a prime number.
4. Group the prime numbers into pairs.
5. List the number from each pair only once outside of the radicand.
6. Leave any unpaired numbers inside the radical.
Note: If you have more than one pair, multiply the numbers outside of the radical as well as the numbers left inside.</td><td>Example: $\sqrt{75}$

$5\sqrt{3}$</td></tr>
</table>

<table>
<tr><td>Method 3:
Divide by Prime Numbers
1. Work with only the radicand.
2. Using only prime numbers, divide each radicand until the bottom number is a prime number.
3. Group the prime numbers into pairs.
4. List the number from each pair only once outside of the radicand.
5. Leave any unpaired numbers inside the radical.
Note: If you have more than one pair, multiply the numbers outside of the radical as well as the numbers left inside.</td><td>Example: $\sqrt{75}$

$5\sqrt{3}$</td></tr>
</table>

Method 4: Use Exponent Rules	**Example:** $\sqrt{75}$
1. Rewrite the exponent as a rational exponent.	$75^{1/2}$
2. Rewrite the radicand as factors that are perfect squares and factors that are not perfect squares.	$(25\cdot 3)^{1/2}$
3. Rewrite the perfect square factors with an exponent of 2.	$(5^2\cdot 3)^{1/2}$
4. Split up the factors, giving each the rational exponent.	$(5^2)^{1/2}\cdot(3)^{1/2}$
5. Simplify.	$5\cdot(3)^{1/2}$
6. Rewrite as a radical	$5\sqrt{3}$

Method 4 with Variables:	**Example:**
	$\sqrt[3]{x^7}$
1. Rewrite the exponent as a rational exponent.	$x^{7/3}$
2. Rewrite the radicand as two factors. One with the highest exponent that is divisible by the root and the other factor with an exponent of what is left over.	$(x^6\cdot x)^{1/3}$
3. Split up the factors, giving each the rational exponent.	$(x^6)^{1/3}\cdot x^{1/3}$
4. Rewrite the exponents using exponent rules.	$x^{6/3}\cdot x^{1/3}$
5. Simplify.	$x^2\cdot x^{1/3}$
6. Rewrite as a radical	$x^2\ \sqrt[3]{x}$

Sample Questions:

30. If x is a real number such that $x^3 = 729$, then $x^2 + \sqrt{x} = ?$

31. Which integer is nearest to $\frac{\sqrt{2100}}{\sqrt{7}}$?

32. If x > 1, then which of the following has the LEAST value?
   a. $\sqrt{x}$
   b. $\sqrt{2x}$
   c. $\sqrt{x * x}$
   d. $x\sqrt{x}$
   e. x*x

33. $\sqrt{27}$ + $\sqrt{48}$ = ?

34. If $\sqrt{2x}$ + 5 = 9, then x = ?

35. What is the smallest integer greater than $\sqrt{58}$?

36. If x is a positive real number such that $x^2 = 16$, then $x^3 + \sqrt{x}$ = ?

37. $\frac{4}{\sqrt{2}} + \frac{2}{\sqrt{3}}$ = ?

38. For positive real numbers x, y, and z, which of the following expressions is equivalent to $x^{1/2}y^{3/4}z^{5/8}$?
   a. $\sqrt[4]{xy^3z^5}$
   b. $\sqrt[8]{x^2y^3z^5}$
   c. $\sqrt[8]{x^4y^3z^5}$
   d. $\sqrt[8]{x^2y^6z^5}$
   e. $\sqrt[14]{xy^3z^5}$

REMOVE RADICALS FROM DEMONINATOR

If a radical is in the denominator of a fraction, simplify to get it out. If the fraction is unable to simplify to remove the radical in the denominator, multiply the top and bottom of the fraction by the radical to alleviate the problem.

Example: $\frac{\sqrt{15}}{\sqrt{12}}$

Solution: Write both roots in terms of their factors, separate the roots. Since the square root of 3 divided by itself is 1. Since the square root of 4 is equal to 2, we have simplified to remove the root from the denominator.

$\frac{\sqrt{15}}{\sqrt{12}} = \frac{\sqrt{5}*\sqrt{3}}{\sqrt{4}*\sqrt{3}} = \frac{\sqrt{5}*\sqrt{3}}{\sqrt{4}*\sqrt{3}} = \frac{\sqrt{5}}{\sqrt{4}} = \frac{\sqrt{5}}{2}$

Example: Simplify $\frac{\sqrt{4}}{4\sqrt{5}}$

Solution: The square root of 4 is equal to 2 and 2 divided by 4 is equal to ½. This simplifies the fraction, but there is still a square root of 5 on the bottom. So, multiply both the top and bottom of the fraction by the square root of 5.

$\frac{\sqrt{4}}{4\sqrt{5}} = \frac{2}{4\sqrt{5}} = \frac{1}{2\sqrt{5}} * \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{2\sqrt{5}*\sqrt{5}} = \frac{\sqrt{5}}{2*5} = \frac{\sqrt{5}}{10}$

Sample Questions:

39. Simplify $\frac{4\sqrt{2}}{3\sqrt{5}}$

40. Simplify $\frac{\sqrt[5]{12}}{4\sqrt[5]{4}}$

41. Simplify $\frac{\sqrt[3]{10}}{\sqrt[3]{625}}$

42. Simplify $\frac{3\sqrt[4]{4}}{2\sqrt[4]{8}}$

SOLVING EQUATIONS WITH EXPONENTS & ROOTS
Use algebra and the properties of exponents and roots to solve algebraic equations.

Example: $27 = x^{3/2}$
Solution: Remember that $x^{\frac{m}{n}} = \sqrt[n]{x^m}$ so the variable part can be written as $\sqrt[2]{x^3}$ so the equation can be rewritten as $27 = \sqrt[2]{x^3}$. To solve square both sides of the equation and then take the cube root of both sides of the equation.
$27 = \sqrt[2]{x^3}$
$(27)^2 = (\sqrt[2]{x^3})^2$
$729 = x^3$
$\sqrt[3]{729} = \sqrt[3]{x^3}$
$9 = x$

Example: $3 = \sqrt{x-1}$
Solution: First square both sides of the equation and then add 1 to both sides.
$3 = \sqrt{x-1}$
$3^2 = (\sqrt{x-1})^2$
$9 = x - 1$
$+1 \quad +1$
$10 = x$

Sample Questions:

43. $a^{5/4} = 243$

44. $7 = x^{1/2}$

45. $2 = \sqrt{\frac{x}{2}}$

46. $\sqrt{-8-2x} = 0$

COMPARING EQUATIONS (EQUAL TO)

When comparing two problems, set part of the equations to be the same, then set the rest of the equation to be equal and solve. Look at the problem to determine similarities. In many problems, set the base to be the same.

Example: If $9^{2x-1} = 3^{3x+3}$, then x = ?

a. -4
b. -7/4
c. -10/7
d. 2
e. 5

Solution: Express the left side of the equation so that both sides have the same base:

$9^{2x-1} = 3^{3x+3}$
$(3^2)^{2x-1} = 3^{3x+3}$
$3^{4x-2} = 3^{3x+3}$

Now that the bases are the same, just set the exponents equal:

$4x - 2 = 3x + 3$
$4x - 3 = 3 + 2$
$X = 5$ or E

Other problems will have similar numerators or denominators. Make them exactly the same and set the rest of the equation to be equal.

Example: What is the sum of all solutions to $\frac{4x}{x-1} = \frac{4x}{2x+2}$?

a. -3
b. -2
c. 2
d. 5
e. 8

Solution: Since the numerators are both 4x, set the denominators equal to each other

$x - 1 = 2x + 2$
$\underline{-x - 2 \quad -x - 2}$
$-3 = x$ or A

Sample Questions:

47. In real numbers, what is the solution of the equation $8^{2x+1} = 4^{1-x}$?

48. Which real number satisfies $(2^n)(8) = 16^3$?

49. If $3^{8x} = 81^{3x-2}$, what is the value of x?

50. If a, b, and c or consecutive positive integers and $2^a \times 2^b \times 2^c = 512$, then a + b + c = ?

51. If $\frac{4\sqrt{9}}{y\sqrt{11}} = \frac{4\sqrt{9}}{11}$, then y = ?

EVEN & ODD VARIATIONS

Sample Questions:

52. If m, n, and p are positive integers such that m + n is even and the value of $(m + n)^2 + n + p$ is odd, which of the following must be true?
    a. m is odd
    b. n is even
    c. p is odd
    d. If n is even, p is odd
    e. If p is odd, n is odd

53. Which of the following is true for all consecutive integers *m* and *n* such that $m < n$?
    a. *m* is odd
    b. *n* is odd
    c. $n - m$ is even
    d. $n^2 - m^2$ is odd
    e. $m^2 + n^2$ is even

Answer Key

1. $x^3 + 7x + 5$
2. $3x^2 + x - 2$
3. $-56x^9 + 24x^8$
4. $6x^9y^9$
5. x^{-6}
6. $\frac{x^4y^9}{z^2}$
7. $27x^9$
8. y
9. -21
10. $3r^2$
11. $243x^{15}$
12. $6r^3$
13. $\frac{1}{2x^{37/12}}$
14. $3y^{17/12}$
15. $\frac{x^9}{y^9}$
16. x – y = 2
17. 19
18. $9\sqrt{5}$
19. 6
20. $12\sqrt{3} - 9\sqrt{15}$
21. $3\sqrt{2}$
22. $3\sqrt[3]{6}$
23. $2xy^2\sqrt[3]{2y^2}$
24. $3y\sqrt[4]{5x^3y}$
25. $2x\sqrt[5]{7x^2}$
26. $(\sqrt{10y})^3$
27. $(6r)^{1/2}$
28. $\frac{1}{\sqrt{5m}}$
29. $2^{5/4}$
30. 84
31. 17
32. $\sqrt{x}$
33. $7\sqrt{3}$
34. 8
35. 8
36. 66
37. $\frac{4\sqrt{3}+2\sqrt{2}}{\sqrt{6}}$
38. $\sqrt[8]{x^4y^6z^5}$
39. $\frac{4\sqrt{10}}{15}$
40. $\frac{\sqrt[5]{3}}{4}$
41. $\frac{\sqrt[3]{2}}{5}$
42. $\frac{3\sqrt[4]{8}}{4}$
43. 81
44. 49
45. 8
46. -4
47. $-\frac{1}{8}$
48. 9
49. 2
50. 9
51. $\sqrt{11}$
52. D
53. D