Math 2 Proportion & Probability Part 3 Sums of Series, Combinations & Compound Probability

SUMMING AN ARITHMETIC SERIES USING A FORMULA

To sum up the terms of this arithmetic sequence:

a + (a+d) + (a+2d) + (a+3d) + ...

$$\sum_{k=0}^{n-1} (a+kd) = \frac{n}{2}(2a+(n-1)d)$$

a is the first term d is the "common difference" between terms n is the number of terms to add up

What is that funny symbol? It is called sigma notion

 \sum (called Sigma)means "sum up"

And below and above it are shown the starting and ending values:

that at this value
go to this value
$$n=1$$
 what to sum
 $n=1+2+3+4=10$

It says "Sum up n where n goes from 1 to 4. Answer=10

Example: Add up the first 10 terms of the arithmetic sequence:

{1, 4, 7, 10, 13, ... } Solution: The values of a, d and n are: a = 1 (the first term) d = 3 (the "common difference" between terms) n = 10 (how many terms to add up) So: $\sum_{k=0}^{n-1} (a + kd) = \frac{n}{2} (2a + (n - 1)d)$ Becomes: $\sum_{k=0}^{10-1} (1 + k \cdot 3) = \frac{10}{2} (2 \cdot 1 + (10 - 1) \cdot 3) = 5(2+9\cdot3) = 5(29) = 145$

Check: Add up the terms yourself using a calculator, and see if it comes to 145

Example: Cynthia decorates the ceiling of her bedroom with stars that glow in the dark. She puts 1 star on the ceiling on the first day of decorating, 2 stars on the ceiling on the 2^{nd} day of decorating, 3 stars on the 3^{rd} day, and so on. If she puts stars on the ceiling in this pattern for 30 days (so she puts 30 stars on the ceiling on the 30^{th} day), then what will be the total number of stars on the ceiling at the end of 30 days?

Solution: You can use your calculator to add 1 + 2 + 3 + 4 + ... + 30 = 465. Or if you identified this as an arithmetic series, you can use the equation:

$$\sum_{k=0}^{n-1} (a + kd) = \frac{n}{2} (2a + (n-1)d)$$

Where a = 1, d = 1, n = 30
Becomes:
$$\sum_{k=0}^{n-1} (1 + k1) = \frac{30}{2} [2(1) + (30 - 1)(1)] = 15 \times 31 = 465$$

Math 2 P&P Part 3

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The simplified equation for finding the sum of arithmetic sequences is:

S = $(a_1 + a_n) \times \frac{n}{2}$ a₁ is the first term a_n is the last term n is the number of terms to add up

Example: Cynthia decorates the ceiling of her bedroom with stars that glow in the dark. She puts 1 star on the ceiling on the first day of decorating, 2 stars on the ceiling on the 2nd day of decorating, 3 stars on the 3rd day, and so on. If she puts stars on the ceiling in this pattern for 30 days (so she puts 30 stars on the ceiling on the 30th day), then what will be the total number of stars on the ceiling at the end of 30 days?

Solution: This time, use the simplified formula $S = (a_1 + a_n) \times \frac{n}{2}$, where n is the number of terms (30), 1a is the first term (1), an is the nth term (30).

$$(1 + 30) \times \frac{30}{2} = 465.$$

Sample Questions:

1. Carmen is playing with blocks. She arranges stacks of blocks so that each successive level of blocks has 1 fewer block than the level below it and the top level has 1 block. Such a stack with 3 levels is shown below. Carmen wants to make such a stack with 12 levels. How many blocks would she use to build this stack?



- 2. A theater has 50 rows of seats. There are 18 seats in the first row, 20 seats in the second, 22 in the third and so on. How many seats are in the theater?
- 3. On the first day of school, Mr. Vilani gave his third-grade students 5 new words to spell. On each day of school after that, he gave the students 3 new words to spell. In the first 20 days of school, how many new words had he given the students to spell?
- 4. The eleventh term of an arithmetic sequence is 30 and the sum of the first eleven terms is 55. What is the common difference?
- 5. How many terms of the arithmetic sequence 2, 8, 14, 20, ... are required to give a sum of 660?

SUMMING AN ARITHMETIC SERIES WITH SLOT FORMULA OR CONSECUTIVE INTEGERS

To solve via slot method, draw appropriate number of slots and figure out how the numbers in the slots are related. Figure out a formula or other solution to the problem and solve. Consecutive integers can be written as n, n+1, n+2, etc. Consecutive odd or even integers can be written as n, n+2, n+4, etc.

Example: The average of 7 consecutive numbers is 16. What is the sum of the least and greatest of the 7 integers?

Solution: n + (n + 1) + (n + 2) + (n + 3) + (n + 4) + (n + 5) + (n + 6) = 16

 $\frac{7n+21}{7} = 16$

Multiply both sides by 7 and subtract 21 from each side.

7n = 91

Divide both sides by 7 and get n = 13

So, the first term is 13. The series is 13, 14, 15, 16, 17, 18, 19. The sum of 13 + 19 = 32

Sample Questions:

6. The 6 consecutive integers below add up to 513.
n - 2
n - 1
n + 1
n + 2
n + 3
What is the value of n?

7. What is the smallest possible value for the product of 2 real numbers that differ by 6?

- 8. The product of two consecutive integers is between 137 and 159. Which of the following CAN be one of the integers?
 - a. 15
 - b. 13
 - c. 11
 - d. 10
 - e. 7

SUMMING A GEOMETRIC SERIES

When we need to sum a Geometric Sequence, there is a handy formula.

To sum: $a + ar + ar^2 + ... + ar^{n-1}$

Each term is ark, where k starts at 0 and goes up to n-1 Use this formula:

$$\sum_{k=0}^{n-1} (ar^k) = a\left(\frac{1-r^n}{1-r}\right)$$

a is the first term r is the "common ratio" between terms n is the number of terms The formula is easy to use ... just "plug in" the values of a, r and n.

Example: Sum the first 4 terms of

10, 30, 90, 270, 810, 2430, . . .

This sequence has a factor of 3 between each number.

The values of a, r and n are:

a = 10 (the first term)

r = 3 (the "common ratio")

n = 4 (we want to sum the first 4 terms)

So:

$$\sum_{k=0}^{n-1} (ar^k) = a\left(\frac{1-r^n}{1-r}\right)$$

Becomes:

$$\sum_{k=0}^{4-1} (10 \cdot 3^k) = 10 \left(\frac{1-3^4}{1-3}\right) = 400$$

And, yes, it is easier to just add them in this example, as there are only 4 terms. But imagine adding 50 terms ... then the formula is much easier.

The **simplified equation** for finding the sum of geometric sequences looks very similar. If S_n represents the sum of the first n terms of a geometric series, then S_n = a + ar + ar² + ar³ + ar⁴ + + arⁿ⁻¹ Where a = the first term, r = the common ratio, and the nth term is not arⁿ, but arⁿ⁻¹ The sum of the first n terms is given by S_n where S_n = $\frac{a (1-r^n)}{1-r}$ The sum to infinity, written as S_∞ is given by S_∞ = $\frac{a}{1-r}$ only if -1 < r < 1

Example: The sum of an infinite geometric sequence series with first term x and common ratio y < 1 is given by $\frac{x}{(1-y)}$. The sum of a given infinite geometric series is 200, and the common ratio is 0.15. What is the second term of this series?

Solution:
$$S_{\infty} = \frac{a}{1-r}$$

200 = $\frac{x}{1-0.15} = 170$
170 * 0.15 = 25.5

Sample Questions:

9. Find the sum of each of the geometric series $-2, \frac{1}{2}, -\frac{1}{8}, \dots, -\frac{1}{37268}$

10. Work out the sum $\sum_{r=1}^{\infty} \left(\frac{1}{3}\right)^r$

11. Find the sum of the first ten terms of the geometric sequence: an=132(-2)n-1.

12. If the sum of the first seven terms in a geometric series is 2158 and r=-12, find the first term.

MULTI STEP SLOT METHOD AND PROBABILITY PROBLEMS

Some problems are more complicated and require combining several problem solving methods into one problem.

Example: Elias has to select one shirt, one pair of pants, and one pair of shoes. If he selects at random from his 8 shirt, 4 pairs of pants, and 3 pairs of shoes, and all his shirts, pants, and shoes, and all are different colors, what is the likelihood that he will select his red shirt, black pants, and brown shoes?

- a. 1/3
- b. 1/4
- c. 1/15
- d. 1/32
- e. 1/96

Solution:

Step 1: Know the question. The problem is asking what the probability is that he will select this one group of clothes from all possible combinations of clothes.

Step 2: Let the answers help. The answers offer the important reminder that we're looking for a probability. We know that it will be only one arrangement out of a reasonably large number of them, so we should at least get rid of choices (A) and (B).

Step 3: Break the problem into bite-sized pieces. First, we should find the total number of possible combinations. Then, we can deal with the probability.

We have three slots to fill here, and we want to find the produce of the three.

 $\frac{8}{\text{Shirts}} \times \frac{4}{\text{Pants}} \times \frac{3}{\text{Shoes}} = 96$

Of the 96 possible arrangements, an ensemble of red shirt, black pants, and brown shoes is only one. Therefore, we can go to the $\frac{part}{whole}$ ratio to find $\frac{part}{whole} = \frac{red, black, brown}{All arrangements} = \frac{1}{96}$, choice (E).

Example: If 3 people all shake hands with each other, there are a total of 3 handshakes. If 4 people all shake hands with each other, there are a total of 6 handshakes. How many total handshakes will there be if 5 people all shake hands with each other?

Solution:

ABCDE

Now you only need to identify the number of different combinations.

- AB, AC, AD, AE BC, BD, BE
- CD, CE
- DE

There are 10 distinct combinations, meaning 10 total handshakes.

6

Sample Question:

- 13. In a basketball passing drill, 5 basketball players stand evenly spaced around a circle. The player with the ball (the passer) passes it to another player (the receiver). The receiver cannot be the player to the passer's immediate right or left and cannot be the player who last passed the ball. A designated player begins the drill as the first passer. This player will be the receiver for the first time on which pass of the ball?
 - a. 4th
 - b. 5th
 - c. 6th
 - d. 10th
 - e. 24th
- 14. In a geometric sequence, the quotient of any two consecutive terms is the same. If the third term of a geometric sequence is 8 and the fourth term is 16, then what is the second term?

- 15. In a game, 80 marbles numbered 00 through 79 are placed in a box. A player draws 1 marble at random from the box. Without replacing the first marble, the player draws a second marble at random. If both marbles drawn have the same ones digit (that is, both marbles have a number ending in 0, 1, 2, 3, etc.), the player is a winner. If the first marble drawn is numbered 35, what is the probability that the player will be a winner on the next draw?
- 16. Carol has an empty container and puts in 6 red chips. She now wants to put in enough white chips so that the probability of drawing a red chip at random form the container is 3/8. How many white chips should she put in?
- 17. In a game, 84 marbles numbered 00 through 83 are placed in a box. A player draws 1 marble at random from the box. Without replacing the first marble, the player draws a second marble at random. If both marbles drawn have the same tens digit (that is, both marbles are numbered between 00 and 09, or 10 and 19, or 20 and 29, etc.), the player is a winner. If the first marble Dave draws is numbered 23, what is the probability that Dave will be a winner on the next draw?

- 18. An integer from 100 through 999, inclusive, is to be chosen at random. What is the probability that the number chosen will have 0 as at least 1 digit?
- 19. As part of a probability experiment, Elliott is to answer 4 multiple-choice questions. For each question, there are 3 possible answers, only 1 of which is correct. If Elliot randomly and independently answers each question, what is the probability that he will answer the 4 questions correctly?
- 20. For every positive 2-digit number, x, with tens digit t and units digit u, let y be the 2-digit number formed by reversing the digits of x. Which of the following expressions is equivalent to x y?
- 21. For every positive 2-digit number, a, with units digit x and tens digit y, let b be the 2-digit number formed by reversing the digits of a. Which of the following expressions is equivalent to a b?



22. The blood types of 150 people were determined for a study as shown in the figure below.

If 1 person from this study is randomly selected, what is the probability that this person has either Type A or Type AB blood?

TYPES OF DATA AND TABLES

Variables that take on values that are names or labels are considered categorical data. The color of a ball (e.g., red, green, blue), gender (male or female), year in school (freshman, sophomore, junior, or senior). These are data that cannot be averaged or represented by a scatter plot as they have no numerical meaning.

Variables that represent a measurable quantity are numerical or quantitative data. For example, when looking at the population of the city, it is the number of people in the city that is the area of interest and this is a measurable attribute of the city. Other examples include scores on a set of tests, height and weight, and temperature at the top of each hour.

Example: Answer the question. If it can't be answered, explain why. The temperature at the park over a 12-hour period was: 60, 64, 66, 71, 75, 77, 78, 80, 78, 77, 73, and 65. Find the average temperature over the 12-hour period.

Solution: The data is quantitative so it is possible to find the average of 72

Example: Once a week, Maria has to fill her car up with gas. She records the day of the week each time she fills her car for three months. Monday, Friday, Tuesday, Thursday, Monday, Wednesday, Monday, Tuesday, Wednesday, Tuesday, and Thursday. Find the average day of the week that Maria had to fill her car with gas. **Solution:** The data is categorical so the average cannot be calculated. However, it is clear that Monday was the day that she most often filled her car.

Sample Questions:

- 23. A group of students has ages of 12, 15, 11, 14, 12, 11, 15, 13, 12, 12, 11, 14, and 13. Find the average age of the students.
- 24. Suppose 11-year-olds are in 6th grade, 12-year-olds are in 7th grade, 13-year-olds are in 8th grade, 14-year olds are in 9th grade, and 15-year-olds are in 10th grade. A group of students has ages of 12, 15, 11, 14, 12, 11, 15, 13, 12, 12, 11, 14, and 13. Find the average grade of the students.
- 25. Katie is collecting data about the number of people that are interested in different dance styles. After passing out a survey, she has gathered the following information:Style / Number of people interested in this styleBallet / 20

Jazz /12 Hip Hop / 8 Lyrical / 26 Tap / 10 Find the average dance interest of the people surveyed.

TWO-WAY FREQUENCE TABLE

A two-way frequency table is a useful tool for examining the relationships between categorical variables. The entries in the cells of a two-way table can be frequency counts or relative frequencies. The table summarizes and shows how often a value occurs.

Generally, conditional relative frequencies are written as a decimal or percentage. It is the ratio of the observed number of a particular event to the total number of events, often taken as an estimate of probability. You can use the relative frequency to determine how often a value may occur in the future.

In a two-way table, entries in the "total" row and "total" columns are called marginal frequencies. Entries in the body of the table are called joint frequencies.

The two-way frequency table shows the favorite leisure activities for 20 men and 30 women using frequency counts.

	Dance	Sports	Movies	Total
Women	16	6	8	30
Men	2	10	8	20
Total	18	16	16	50

The two-way frequency table shows the same information represented as conditional relative frequencies.

	Dance	Sports	Movies	Total
Women	0.32	0.12	0.16	0.60
Men	0.04	0.20	0.16	0.40
Total	0.36	0.32	0.32	1.00

Two-way tables can show relative frequencies for the whole table, for rows, or for columns.

Relative Frequency of Rows				Re	lative Fr	equency	of Colum	ns			
		Dance	Sports	Movies	Total			Dance	Sports	Movies	Total
	Women	0.53	0.20	0.27	1.00	Won	ıen	0.89	0.38	0.50	0.60
	Men	0.10	0.50	0.40	1.00	Me	n	0.11	0.62	0.50	0.40
	Total	0.36	0.32	0.32	1.00	Tot	al	1.00	1.00	1.00	1.00

Each type of relative frequency table makes a different contribution to understanding the relationship between gender and preferences for leisure activity. For example, the "Relative Frequency of Rows" table most clearly shows the probability that each gender will prefer a particular leisure activity. For instance, it is easy to see that the probability that a man will prefer movies is 40% while it is 27% for women. **Example:** A public opinion survey explored the relationship between age and support for increasing the minimum wage. The results are found in the following two-way frequency table.

	For	Against	No Opinion	TOTAL
Ages 21 – 40	25	20	5	50
Ages 41 – 60	30	30	15	75
Over 60	50	20	5	75
TOTAL	105	70	25	200

In the 41 to 60 age group, what percentage supports increasing the minimum wage? Answer: A total of 75 people in the 41 to 60 age group were surveyed. Of those, 30 were for increasing the minimum wage. Thus, 40% supported increasing the minimum wage.

Example: Create two-way frequency tables to organize the information from the situations described. A bank teller splits transactions into two categories: deposits and withdrawals. In one shift, the bank teller has 72 transactions. Of those, 12 males make deposits and 30 make withdrawals. While 20 females make deposits. **Solution:** First, draw an empty frequency table and add the appropriate labels.

	Deposits	Withdrawals	Total
Men			
Women			
Total			

Then add the numbers that are listed in the question to the table.

	Deposits	Withdrawals	Total
Men	12	30	
Women	20		
Total			72

Finally, calculate the numbers that are missing and add them to the table. Verify that all columns and rows add up correctly.

Total Deposits: 12 + 20 = 32

To get the total transactions to be 72, there must be 72 - 32 = 40 withdrawals.

To get a total of 40 withdrawals, there must be 40 - 30 = 10 by women

That makes a total of 20 + 10 = 30 transactions by women.

Total transactions by men = 12 + 30 = 42

If you add the total transactions by men and women, you get 42 + 30 = 72

Which is verified by adding the total deposits and withdrawals 32 + 40 = 72

	Deposits	Withdrawals	Total
Men	12	30	42
Women	20	10	30
Total	32	40	72

Sample Questions:

Create two-way frequency tables to organize the information from the situations described.

26. Heather has a dance studio that offers classes in both contemporary and hip-hop dance. She has 38 female hip-hop dancers and 43 male hip-hop dancers. Heather has a total of 200 dancers enrolled in classes with 60 of them being male.

27. Sarah is worried about how much garbage she creates each week. She decides to look at how many items she could recycle in three weeks' time. She has cans, glass bottles, and newspapers that can be recycled. She was able to recycle 5 cans, then 3 cans, and finally 4 cans in the third week. She recycled 6 glass bottles every week. During the first two weeks she recycled 2 newspapers and 3 newspapers in the third week.

28. Mr. Smith splits students that did not do their homework into two categories: first timers and second(+) timers. In one month, 36 girls and 12 boys did not do their homework for the first time. Twelve girls and 30 boys did not do their homework for the second or more times.

29. Complete the two-way frequency table and answer the questions about the information in the table

	Male	Female	Total		
Eat Breakfast Regularly		110	300		
Don't Eat Breakfast Regularly	130	165			
Total		275			

Hollywood Junior students were interviewed about their eating habits.

- a. Of the total males, what percentage do not eat breakfast regularly?
- b. Of the total people who eat breakfast regularly, what percentage of them are males?
- c. How many females eat breakfast regularly?
- d. How many females were included in the survey?
- e. How many females eat breakfast out of the total number of females?
- f. How many people were included in this survey?
- g. How many males do not eat breakfast regularly?
- h. How many males and females do not eat breakfast regularly?
- i. Which group of people eat breakfast more regularly?

30. Complete the two-way frequency table and answer the questions about the information in the table.

	Male	Female	Total
Walk		46	
Car	28		45
Bus		12	27
Cycle		17	69
Total	129	92	

Jersey High students were surveyed about the type of transportation they use the most often.

- a. Of the total males, what percentage cycle?
- b. Of the total people who use the bus, what percentage are female?
- c. How many more females than males walk?
- d. What is the percent of males that walk compared to ride something?
- e. How many females ride in a car?
- f. How many total people surveyed?
- g. How many total males cycle?

VENN DIAGRAMS

An event is an activity or experiment which is usually represented by a capital letter. A sample space is a set of all possible outcomes for an activity or experiment. A smaller set of outcomes from the sample space is called a subset. The complement of a subset is all outcomes in the sample space that are not part of the subset. A subset and its complement make up the entire sample space. If a subset is represented by A, the complement can be represented by any of the following: not A, ~A, or A^c .

Event	Sample Space	Possible Subset	Complement
Flip a coin	S={heads, tails}	B={heads}	$\mathbf{B}^{c} = \{ \text{tails} \}$
Roll a die	S={1, 2, 3, 4, 5, 6}	even E={2, 4, 6}	~E ={1, 3, 5}
Pick a digit 0-9	$S=\{0,1, 2, 3, 4, 5, 6, 7, 8, 9\}$	N={2, 5, 7, 9}	not N= {0, 1, 3, 4, 6, 8}

Definition	Example	Venn Diagram
The union of two events includes all outcomes from each event. The union can be indicated by the word "or" or the symbol \bigcup .	$A = \{0, 2, 4, 6, 8, 10\}$ B= {0, 5, 10, 15, 20} A \cup B= {0, 2, 4, 5, 6, 8, 10, 15, 20}	2, 4, 6, 8 (0, 10) 5, 15, 20
The intersection of two events includes only those outcomes that are in both events. The intersection can be indicated by the word "and" or the symbol \bigcirc . If the two events do NOT have anything in common, the intersection is the "empty set", indicated by { } or \emptyset .	$A = \{0, 2, 4, 6, 8, 10\}$ B= {0, 5, 10, 15, 20} A \cap B = {0, 10}	



Sample Questions:

- 31. Given all the letters of the English Alphabet
 - a. Draw a Venn Diagram of the following:
 - i. List a subset of the letters in your first name.
 - ii. List a subset of the letters in your last name.
 - b. Find the union of the subsets of your first name and last name.
 - c. Find the intersection of the subsets of your first name and last name.

- 32. Given the sample space S = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} with event A = {3, 4, 5, 6, 7} and event B = {1, 2, 3, 4, 5}.
 - a. Draw a Venn diagram representing the sample space with events A and B.
 - b. List all the outcomes for AUB.
 - c. List all the outcomes for $A \cap B$.

- 33. Given a standard deck of 52 cards, event A is defined as a red card and event B is defined as the card is a diamond.
 - a. List all the outcomes for $A \cup B$.
 - b. List all the outcomes for $A \cap B$.
 - c. What is $A \cap A^c$? (A^c is the compliment or opposite data of A)

		-
1.	78	12.40
2.	3,350	13. 5 th
3.	62 words	14.4
4.	Common Difference = 5	15. 7/79
5.	15 terms	16. 10 white chips
6.	85	17.9/83
7.	-9	18. 171/900
8.	13	19. 1/81
9.	5/4	20. 9(t – u)
10	. S∞ = 1/2	21. 9(x – y)
11.		22.73/150
	3/	

23. The data is quantitative so it is possible to find the average age of 12.69.

- 24. Even though the grade level is a number the data is categorical so the average grade level cannot be found. However, is clear that 7th grade has the highest number of students.
- 25. Even though there are a number of people that like each type of dance, the data is categorical so the average dance interest cannot be found. However, is clear that the most people surveyed like lyrical dance.

26.

	Contemporary	Hip-hop	Total
Female	102	38	140
Male	17	43	60
Total	119	81	200

27.

	Cans	Glass Bottles	Newspapers	Total
Week 1	5	6	2	13
Week 2	3	6	2	11
Week 3	4	6	3	13
Total	12	18	7	37

28.

	First Timers	Second Timers	Total
Female	36	12	48
Male	12	30	42
Total	48	42	90

29.

	Male	Female	Total
Eat Breakfast Regularly	190	110	300
Don't Eat Breakfast Regularly	130	165	295
Total	320	275	595
a. 41%	d. 300		g. 130
b. 63%	e. 110		h. 320
c. 110	f. 595		i. male

	Male	Female	Total
Walk		46	
Car	28		45
Bus		12	27
Cycle		17	69
Total	129	92	

- a. 40%
- b. 44%c. 12

d. 36% e. 17

f. 221

g. 52

31. Example: Jennifer Wright



b. Union: E, F, G, H, I, J, N, R, T, W

c. Intersection: R, I

32.





AS, 2S, 3S, 4S, 5S, 6S, 7S, 8S, 9S, JS, QS, KS, AC, 2C, 3C, 4D, 5C, 6C, 7C, 8C, 9C, JC, QC, KC

- a. AH, 2H, 3H, 4H, 5H, 6H, 7H, 8H, 9H, JH, QH, KH, AD, 2D, 3D, 4D, 5D, 6D, 7D, 8D, 9D, JD, QD, KD
- b. AD, 2D, 3D, 4D, 5D, 6D, 7D, 8D, 9D, JD, QD, KD
- c. AD, 2D, 3D, 4D, 5D, 6D, 7D, 8D, 9D, JD, QD, KD, AS, 2S, 3S, 4S, 5S, 6S, 7S, 8S, 9S, JS, QS, KS, AC, 2C, 3C, 4D, 5C, 6C, 7C, 8C, 9C, JC, QC, KC
- d. NONE