

## 8<sup>th</sup> Grade Honors - Proportions & Probability Part 3

### COMBINED PERCENT INCREASE AND DECREASE

When there are multiple percent increases and/or decreases, and the question asks for the combined percent increase or decrease, the easiest and most effective strategy is to pick 100 for the original value and see what happens.

**Example:** A price went up 10% one year, and the new price went up 20% the next year. What was the combined percent increase?

**Solution:** First year:  $100 + (10\% \text{ of } 100) = 110$ . Second year:  $110 + (20\% \text{ of } 110) = 132$ .

That's a combined 32% increase.

Sample Questions:

1. A shoe store charges \$39 for a certain type of sneaker. This price is 40% more than the amount it costs the shoe store to buy one pair of these sneakers. At an end-of-the-year sale, sales associates can purchase any remaining sneakers at 20% off the shoe store's cost. How much would it cost an employee to purchase a pair of sneakers of this type during the sale (excluding sales tax)?
2. On Friday, a computer was priced at \$800. On the following Wednesday, the price was reduced by 15%. On the following Friday, the price was further reduced by 20%. What percent of the original price was the final price?
3. A number is increased by 25% and the resulting number is then decreased by 20%. The final number is what percent of the original number

### PART-TO-PART AND PART-TO-WHOLE RATIOS

A part-to-part ratio can be turned into two part-to-whole ratios by putting **each number in the original ratio over the sum of the parts**. If the ratio of males to females is 1 to 2, then the males-to-people ratio is  $\frac{1}{3}$  and the females-to-people ratio is  $\frac{2}{3}$ . Or,  $\frac{1}{3}$  of all the people are male.

Sample Questions:

4. Esther is making  $2\frac{1}{4}$  gallons of punch for a large party. While mixing the punch, she uses  $\frac{1}{2}$  gallon of pineapple juice. What fraction of the punch consists of pineapple juice?
5. In a poll, 44 people were in favor of constructing a new high school, 58 were against it, and 8 people had no opinion. What fraction of those people polled were in favor of constructing a new high school?

## MULTI STEP RATIO PROBLEMS

The most common ratio problems involve a comparison between two quantities. These ratios are called two-term ratios. There are three basic steps you must do first, when working any ratio problems: Change the quantities to the same units; then reduce the ratio to its simplest form.

**Example:** What is the ratio of 6 minutes to 8 hours?

**Solution:** First, change the hours to minutes: 8 hours =  $8 \times 60 = 480$  minutes

Write the ratio as a fraction and simplify:

$$\frac{6}{480} = \frac{1}{80}$$

The ratio of 6 minutes to 8 hours is 1:80.

**Remember:** Make sure that there are the same items in the numerator and denominator.

**Example:** The ratio of Olga's classical CDs to her rock CDs is 14 to 25

**Solution:**

$$\frac{\text{classical}}{\text{rock}} = \frac{14}{25}$$

This setup is wrong:

$$\frac{\text{classical}}{\text{rock}} = \frac{25}{14}$$

**Example:** A room is 16 feet, 8 inches long, and the ratio of the length to the width is 4 to 5. What is the width of the room?

**Solution:** Step 1: Since the length is given in both feet and inches, let's convert it to inches using the fact that 1 foot equals 12 inches. To find how many inches are in 16 feet, we multiply 16 feet by 12 inches:

$$16 \text{ feet, } 8 \text{ inches} = (16 \times 12) + 8 = 192 + 8 = 200$$

We found that the length is 200 inches.

Step 2: Let  $x$  represent the width. We can now set up the equation:

$$\frac{\text{width}}{\text{length}} = \frac{4}{5} = \frac{x}{200}$$

$x = 160$  inches The width is 160 inches.

Let's now convert inches to feet so that the units for the width are consistent with the units for the length.

Since 1 foot is 12 inches, we divide 160 inches by 12 to find out how many feet are in 160 inches:

$160:12 = 13$ , with the remainder of 4 inches. The width is 13 feet, 4 inches. The room width is 13 feet, 4 inches.

Sample Questions:

6. A school has 300 students. If the ratio of boys to girls is 31 to 44, how many more girls are there in the school?

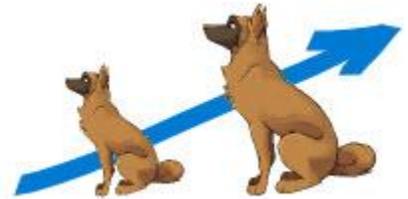
7. At a small college, the ratio of men to women is 9:4. If there are presently 720 women, how many men are at the college?

8. How many additional women would it take to reduce the ratio of men to women to 2:1?
9. The ratio of x to z is 3 to 5, and the ratio of y to z is 1 to 5. What is the ratio of x to y?
10. If  $x:y = 5:2$  and  $y:z = 3:2$ , what is the ratio of  $x:z$  ?

### DIRECTLY PROPORTIONAL AND INVERSELY PROPORTIONAL

#### Directly proportional

Directly proportional means as one amount increases, another amount increases at the same rate. The symbol for "directly proportional" is  $\propto$  (Don't confuse it with the symbol for infinity )



**Example:** You are paid \$20 an hour. How much you earn is directly proportional to how many hours you work. Work more hours, get more pay; in direct proportion.

This could be written:

Earnings  $\propto$  Hours worked

If you work 2 hours you get paid \$40

If you work 3 hours you get paid \$60 etc ...

#### Constant of Proportionality

The "constant of proportionality" is the value that relates the two amounts

**Example:** You are paid \$20 an hour, what is the constant of proportionality?

The constant of proportionality is 20 because: Earnings =  $20 \times$  Hours worked

This can be written:  $y = kx$

Where k is the constant of proportionality

**Example:** y is directly proportional to x, and when  $x=3$  then  $y=15$ . What is the constant of proportionality?

**Solution:** They are directly proportional, so:  $y = kx$

Put in what we know ( $y=15$  and  $x=3$ ):  $15 = k \times 3$   $k = 5$

The constant of proportionality is 5:  $y = 5x$

When we know the constant of proportionality we can then answer other questions.

**Example:** What is the value of y when  $x = 9$ ?

**Solution:**  $y = 5 \times 9 = 45$

**Example:** What is the value of x when  $y = 2$ ?

**Solution:**  $2 = 5x$   $x = 2/5 = 0.4$

### Inversely Proportional

Inversely Proportional: when one value decreases at the same rate that the other increases.

**Example:** speed and travel time

Speed and travel time are Inversely Proportional because the faster we go the shorter the time. As speed goes up, travel time goes down. And as speed goes down, travel time goes up

This is the same thing as:  $y$  is inversely proportional to  $x$  and  $y$  is directly proportional to  $1/x$ . Which can be written  $y =$

**Example:** 4 people can paint a fence in 3 hours. How long will it take 6 people to paint it? (Assume everyone works at the same rate) It is an Inverse Proportion: As the number of people goes up, the painting time goes down. As the number of people goes down, the painting time goes up.

We can use:  $t = k/n$

Where:  $t$  = number of hours,  $k$  = constant of proportionality,  $n$  = number of people

"4 people can paint a fence in 3 hours" means that  $t = 3$  when  $n = 4$

$3 = k/4$  so  $k = 12$

So now we know:  $t = 12/n$

And when  $n = 6$ :  $t = 12/6 = 2$  hours So 6 people will take 2 hours to paint the fence.

**Example:** How many people are needed to complete the job in half an hour?

$\frac{1}{2} = 12/n$   $n = 12 / \frac{1}{2} = 24$  So it needs 24 people to complete the job in half an hour. (Assuming they don't all get in each other's way!)

### Proportional to ...

It is also possible to be proportional to a square, a cube, an exponential, or other function!

**Example:** A stone is dropped from the top of a high tower. The distance it falls is proportional to the square of the time of fall. The stone falls 19.6 m after 2 seconds, how far does it fall after 3 seconds?

We can use:  $d = kt^2$  Where:  $d$  is the distance fallen and  $t$  is the time of fall

When  $d = 19.6$  then  $t = 2$   $19.6 = k \times 2^2$   $19.6 = 4k$   $k = 4.9$  So now we know:  $d = 4.9t^2$

And when  $t = 3$ :  $d = 4.9 \times 3^2$   $d = 44.1$  So it has fallen 44.1 m after 3 seconds.

Sample Questions:

11. If  $c$  is directly proportional to  $s^2$  and  $c =$  when  $s =$  , what is the value of  $s$  when  $c = 175$ ?

12. If  $a$  is inversely proportional to  $b$  and  $a = 36$  when  $b = 12$ , what is the value of  $a$  when  $b = 48$ ?

13. If  $y$  is directly proportional to  $x$  and given  $y = 9$  when  $x = 5$ , find:

a) the value of  $y$  when  $x = 15$

b) the value of  $x$  when  $y = 6$

14. Jane ran 100 meters in 15 seconds. How long did she take to run 2 meter?

15. A car travels 125 miles in 3 hours. How far would it travel in 5 hours?

### RATE PROBLEM USING VARIABLE MANIPULATION

**Example:** A passenger train leaves the train depot 2 hours after a freight train left the same depot. The freight train is traveling 20 mph slower than the passenger train. Find the rate of each train, if the passenger train overtakes the freight train in three hours.

**Solution:** First, interpret the word problem and make a table:

The passenger train drove for three hours to catch up to the freight train; "3". But note that the freight train had a two-hour head start. That means that the freight train was going for five hours.

	d	r	t
passenger train	$d = 3r$	$r$	3
freight train	$d = 5(r - 20)$	$r - 20$	$3 + 2 = 5$
total	---	---	---

Now that I have this information, I can try to find my equation. Using the fact that  $d = rt$ , the first row gives:  $d = 3r$ . The second row gives:  $d = 5(r - 20)$

Since the distances are equal, set the equations equal:  $3r = 5(r - 20)$

Solving gives  $r = 50$ —passenger train is going 50 mph

That means the freight train is going  $50 - 20 = 30$  mph

Sample Problems:

16. A car and a bus set out at 2 p.m. from the same point, headed in the same direction. The average speed of the car is 30 mph slower than twice the speed of the bus. In two hours, the car is 20 miles ahead of the bus. Find the rate of the car.

17. A dog eats 7 cans of food in 3 days. At this rate, how many cans of food does the dog eat in  $3 + d$  days?

18. Harry is a piano student who can learn 2 new pieces of music in a week. If his piano teacher gives him 3 new pieces every week for 4 weeks, how many weeks will it take Harry to learn all these pieces?

19. Vehicle A averages 14 miles per gallon of gasoline, and Vehicle B averages 36 miles per gallon of gasoline. At these rates, how many more gallons of gasoline does Vehicle A need than Vehicle B to make a 1,008-mile trip?
20. A hot-air balloon 70 meters above the ground is falling at a constant rate of 60 meters per second while another hot-air balloon 10 meters above the ground is rising at a constant rate of 15 meters per second. To the nearest tenth of a second, after how many seconds will the 2 balloons be the same height above the ground?

### RATE PROBLEMS WITH DATA

21. The shipping rate for customers of Ship Quick consists of a fee per box and a price per pound for each box. The table below gives the fee and the price per pound for customers shipping boxes of various weights.

Weight of box (pounds)	Fee	Price per pound
Less than 10	5.00	\$1.00
10–25	10.00	\$0.65
More than 25	20.00	\$0.30

Gregg wants Ship Quick to ship 1 box that weighs 15 pounds. What is the shipping rate for this box?

22. Kaylee is planning to purchase a car. She will need to borrow some of the money and has a chart, shown below, to use to approximate her monthly payment. The chart gives the approximate monthly payment per \$1,000 borrowed.

Monthly payment per \$1,000 borrowed for various annual rates and various numbers of payments			
Annual interest rate	Number of monthly payments		
	36	48	60
5%	\$29.97	\$23.03	\$18.87
8%	\$31.34	\$24.41	\$20.28
10%	\$32.27	\$25.36	\$21.24
12%	\$33.22	\$26.34	\$22.24

A local dealership is having an end-of-the-model-year clearance sale and is offering 5% annual interest on new-car loans for 36, 48, or 60 months. The maximum amount Kaylee can budget for her monthly car payment is \$300. Of the following loan amounts, which one is the maximum Kaylee can borrow at 5% annual interest and stay within her budget?

## AVERAGE RATE

Average rate is *not* simply the average of the rates. It's the average of the total amounts. The most common rate is speed-distance over time-and the most common question about average rates is average speed-total distance over total time.

Average A per B =

Average speed =

**Example:** A 555-mile, 5-hour plane trip was flown at two speeds. For the first part of the trip, the average speed was 105 mph. Then the tailwind picked up, and the remainder of the trip was flown at an average speed of 115 mph. For how long did the plane fly at each speed?

**Solution:** First, set up a grid:

	d	r	t
first part	d	105	t
second part	555 - d	115	5 - t
total	555	---	5

Using " $d = rt$ ", the first row gives me  $d = 105t$  and the second row gives:  $555 - d = 115(5 - t)$

Since the two distances add up to 555, add the two distance expressions, and set their sum equal to the given total:  $555 = 105t + 115(5 - t)$   $2 = t$

According to the grid, "t" stands for the time spent on the first part of the trip, so the answer is "The plane flew for two hours at 105 mph and three hours at 115 mph."

**Remember:** You can add distances and you can add times, but you cannot add rates. Think about it: If you drive 20 mph on one street, and 40 mph on another street, does that mean you averaged 60 mph?

Sample Questions:

23. If the first 120 miles of a journey is at 40 mph and the next 120 miles is at 60 mph, what is the average speed?
24. Marion spent all day on a sightseeing trip in Tuscany. First she boarded the bus which went 15mph through a 30 mile section of the countryside. The bus then stopped for lunch in Florence before continuing on a 3 hour tour of the city's sights at speed of 10mph. Finally, the bus left the city and drove 40 miles straight back to the hotel. Marion arrived back at her hotel exactly 2 hours after leaving Florence. What was the bus's average rate for the entire journey?
25. Tracey ran to the top of a steep hill at an average pace of 6 miles per hour. She took the exact same trail back down. To her relief, the descent was much faster; her average speed rose to 14 miles per hour. If the entire run took Tracey exactly one hour to complete and she did not make any stops, what is the length of trail in miles one way?

## SUMMING AN ARITHMETIC SERIES USING A FORMULA

To sum up the terms of this arithmetic sequence:

$$a + (a+d) + (a+2d) + (a+3d) + \dots$$

$a$  is the first term

$d$  is the "common difference" between terms

$n$  is the number of terms to add up

What is that funny symbol? It is called sigma notation

And below and above it are shown the starting and ending values:

$$\sum_{n=1}^4 n = 1+2+3+4 = 10$$

It says "Sum up  $n$  where  $n$  goes from 1 to 4. Answer=10

**Example:** Add up the first 10 terms of the arithmetic sequence:

$$\{ 1, 4, 7, 10, 13, \dots \}$$

**Solution:** The values of  $a$ ,  $d$  and  $n$  are:

$$a = 1 \text{ (the first term)}$$

$$d = 3 \text{ (the "common difference" between terms)}$$

$$n = 10 \text{ (how many terms to add up)}$$

So:

$$\text{Becomes: } \sum_{k=0}^{10-1} (1 + k \cdot 3) = \frac{10}{2} (2 \cdot 1 + (10 - 1) \cdot 3) = 5(2+9 \cdot 3) = 5(29) = 145$$

**Check:** Add up the terms yourself using a calculator, and see if it comes to 145

**Example:** Cynthia decorates the ceiling of her bedroom with stars that glow in the dark. She puts 1 star on the ceiling on the first day of decorating, 2 stars on the ceiling on the 2<sup>nd</sup> day of decorating, 3 stars on the 3<sup>rd</sup> day, and so on. If she puts stars on the ceiling in this pattern for 30 days (so she puts 30 stars on the ceiling on the 30<sup>th</sup> day), then what will be the total number of stars on the ceiling at the end of 30 days?

**Solution:** You can use your calculator to add  $1 + 2 + 3 + 4 + \dots + 30 = 465$ . Or if you identified this as an arithmetic series, you can use the equation:

$$\text{Where } a = 1, d = 1, n = 30$$

Becomes:

$$= \times 31 = 465$$

The **simplified equation** for finding the sum of arithmetic sequences is:

$$S = (a_1 + a_n) \times$$

$a_1$  is the first term

$a_n$  is the last term

$n$  is the number of terms to add up

**Example:** Cynthia decorates the ceiling of her bedroom with stars that glow in the dark. She puts 1 star on the ceiling on the first day of decorating, 2 stars on the ceiling on the 2<sup>nd</sup> day of decorating, 3 stars on the 3<sup>rd</sup> day, and so on. If she puts stars on the ceiling in this pattern for 30 days (so she puts 30 stars on the ceiling on the 30<sup>th</sup> day), then what will be the total number of stars on the ceiling at the end of 30 days?

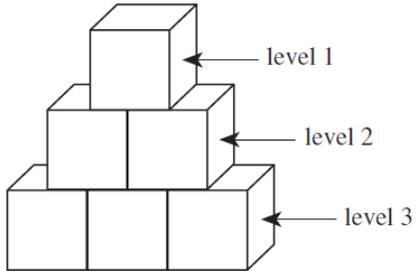
**Solution:** This time, use the simplified formula  $S = (a_1 + a_n) \times n$ , where  $n$  is the number of terms (30),  $a_1$  is the first term (1),  $a_n$  is the  $n$ th term (30).

$$(1 + 30) \times 30 = 465.$$

Sample Questions:

26. The average of 7 consecutive numbers is 16. What is the sum of the least and greatest of the 7 integers?

27. Carmen is playing with blocks. She arranges stacks of blocks so that each successive level of blocks has 1 fewer block than the level below it and the top level has 1 block. Such a stack with 3 levels is shown below. Carmen wants to make such a stack with 12 levels. How many blocks would she use to build this stack?



28. What is the sum of the first 4 terms of the arithmetic sequence in which the 6th term is 8 and the 10th term is 13?

### SUMMING AN ARITHMETIC SERIES USING SLOT FORMULA

Draw appropriate number of slots and figure out how the numbers in the slots are related. Figure out a formula or other solution to the problem and solve.

Sample Questions:

29. The 6 consecutive integers below add up to 513.

$n - 2$

$n - 1$

$n$

$n + 1$

$n + 2$

$n + 3$

What is the value of  $n$ ?

30. What is the smallest possible value for the product of 2 real numbers that differ by 6?

31. The product of two consecutive integers is between 137 and 159. Which of the following CAN be one of the integers?
32. On the first day of school, Mr. Vilani gave his third-grade students 5 new words to spell. On each day of school after that, he gave the students 3 new words to spell. In the first 20 days of school, how many new words had he given the students to spell?

### SUMMING A GEOMETRIC SERIES

When we need to sum a Geometric Sequence, there is a handy formula.

To sum:  $a + ar + ar^2 + \dots + ar^{n-1}$

Each term is  $ar^k$ , where  $k$  starts at 0 and goes up to  $n-1$

Use this formula:

$$\sum_{k=0}^{n-1} (ar^k) = a \left( \frac{1 - r^n}{1 - r} \right)$$

$a$  is the first term

$r$  is the "common ratio" between terms

$n$  is the number of terms

The formula is easy to use ... just "plug in" the values of  $a$ ,  $r$  and  $n$ .

**Example:** Sum the first 4 terms of

10, 30, 90, 270, 810, 2430, . . .

Solution: This sequence has a factor of 3 between each number.

The values of  $a$ ,  $r$  and  $n$  are:

$a = 10$  (the first term)

$r = 3$  (the "common ratio")

$n = 4$  (we want to sum the first 4 terms)

So:

$$\sum_{k=0}^{n-1} (ar^k) = a \left( \frac{1 - r^n}{1 - r} \right)$$

Becomes:

$$\sum_{k=0}^{4-1} (10 \cdot 3^k) = 10 \left( \frac{1 - 3^4}{1 - 3} \right) = 400$$

And, yes, it is easier to just add them in this example, as there are only 4 terms. But imagine adding 50 terms ... then the formula is much easier.

The **simplified equation** for finding the sum of geometric sequences looks very similar. If  $S_n$  represents the sum of the first  $n$  terms of a geometric series, then

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + \dots + ar^{n-1}$$

Where  $a$  = the first term,  $r$  = the common ratio, and the  $n$ th term is not  $ar^n$ , but  $ar^{n-1}$

The sum of the first  $n$  terms is given by  $S_n$  where

$$S_n =$$

The sum to infinity, written as  $S_\infty$  is given by

$$S_\infty = \frac{a}{1-r} \text{ only if } -1 < r < 1$$

Sample Questions:

33. The first and second terms of a geometric sequence are  $a$  and  $ab$ , in that order. What is the 643<sup>rd</sup> term of the sequence?

34. The sum of an infinite geometric sequence series with first term  $x$  and common ratio  $y < 1$  is given by  $\frac{x}{1-y}$ . The sum of a given infinite geometric series is 200, and the common ratio is 0.15. What is the second term of this series?

### MULTI STEP SLOT METHOD PROBLEMS

Some problems are more complicated and require combining several problem solving methods into one problem.

**Example:** Elias has to select one shirt, one pair of pants, and one pair of shoes. If he selects at random from his 8 shirts, 4 pairs of pants, and 3 pairs of shoes, and all his shirts, pants, and shoes, and all are different colors, what is the likelihood that he will select his red shirt, black pants, and brown shoes?

**Solution:**

Step 1: Know the question. The problem is asking what the probability is that he will select this one group of clothes from all possible combinations of clothes.

Step 2: Let the answers help. The answers offer the important reminder that we're looking for a probability. We know that it will be only one arrangement out of a reasonably large number of them, so we should at least get rid of choices (A) and (B).

Step 3: Break the problem into bite-sized pieces. First, we should find the total number of possible combinations. Then, we can deal with the probability.

We have three slots to fill here, and we want to find the produce of the three.

$$\frac{8}{\text{Shirts}} \times \frac{4}{\text{Pants}} \times \frac{3}{\text{Shoes}} = 96$$

Of the 96 possible arrangements, an ensemble of red shirt, black pants, and brown shoes is only one. Therefore, we can go to the ratio to find  $\frac{1}{96}$ , choice (E).

Sample Question:

35. If 3 people all shake hands with each other, there are a total of 3 handshakes. If 4 people all shake hands with each other, there are a total of 6 handshakes. How many total handshakes will there be if 5 people all shake hands with each other?
36. In a basketball passing drill, 5 basketball players stand evenly spaced around a circle. The player with the ball (the passer) passes it to another player (the receiver). The receiver cannot be the player to the passer's immediate right or left and cannot be the player who last passed the ball. A designated player begins the drill as the first passer. This player will be the receiver for the first time on which pass of the ball?
37. In a geometric sequence, the quotient of any two consecutive terms is the same. If the third term of a geometric sequence is 8 and the fourth term is 16, then what is the second term?

## Answer Key

1. \$24.00
2. 68
3. 100%
4.  $\frac{2}{9}$
5.  $\frac{2}{5}$
6. There are 52 more girls than boys in the school.
7. There are presently 1,620 men at the college.
8. 90 additional women
9. 3:1
10. 15:4
11. 5
12. 9
13. A) 27 and B) 3.333
14. 0.3 seconds
15. 20,813 miles
16. 50 mph
17.  $7 + \frac{7}{3}d$
18. 6
19. 44
20. 2.9
21. \$19.75
22. \$15,000
23. 48 mph
24. 14.28 mph
25. 4.2 miles
26. 32
27. 78
28. 39.5
29. 85
30. -9
31. 13
32. 62
33.  $ab^{642}$
34. 25.5
35. 10
36.  $5^{\text{th}}$
37. 4